Update on Control Matrix Studies

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○Motivation:

Our physics optimizer conducts a search in a parameter space $\mathbf{Z} = \{Z_j\}(j = 1, ..N_z)$ describing the stellarator boundary, using an objective function $F(P)$, a function of figures of merit $\mathbf{P} = \{P_i(\mathbf{Z})\}(i = 1, ..M_p)$ characterizing the physics properties (*e.g.*, transport, kink stability, *etc.*) of the configuration. While a powerful tool, the optimizer is searching a space whose topography has been essentially unknown, and we have had rather limited understanding of why the optimizer arrives at the design points $\mathbf{Z}_0$ it does. For example, what are the essential features of the shape that are responsible for the desired physics properties? Deeper insight into this would enhance the our ability to locate attractive design points, and understand the effect of perturbations from those points.

The control matrix (CM) project is intended to provide this insight, through increased understanding of the topography, and through applying mathematical techniques such as SVD methods to gain a clearer grasp of how the $P_i(\mathbf{Z})$ can be changed to achieve superior base configurations and operational flexibility.

∘Outline:

·Formulation
·Some earlier results
·Topography of **Z**-space

·Reducing the dimensionality of **Z**
·Reduced Model & 1st Calculation of G_{ij}, H_{ijk}
·Quadratic Model
·'Proof of Principle' of CM method
·Other QAS Design Points

·Better understanding and focussing the optimizer
·Application to coil design, flexibility & startup issues
·Connection to Quality Matrix (QM) approach

∘Formulation:

∘We consider 2 linearly–related 'configuration spaces' $\mathbf{X}$ and $\mathbf{Z}$ specifying a stellarator:
·The 'full-space' $\mathbf{X}$ of coefficients
$\mathbf{X} \equiv \{X_{j=1,..N_x}\} \equiv (R_{\mathbf{m}_1}, Z_{\mathbf{m}_1}, R_{\mathbf{m}_2}, ...Z_{\mathbf{m}_{N_x/2}})$
needed for (*e.g.*) a VMEC equilibrium specification of the boundary. Here, $\mathbf{m} \equiv (\tilde{n} \equiv n/N_{fp}, m)$, and $N_x \sim 70$.
·The 'reduced-space' $\mathbf{Z}$ of combinations of those X_j which capture the most important physics:
$\mathbf{Z} \equiv \{Z_{j=1,..N_z}\}$, where $N \equiv N_z \leq N_x$.

∘Over $\mathbf{X}$ or $\mathbf{Z}$, we consider the behavior of $M \equiv M_p \sim 5$ physics figures of merit
$\mathbf{P} \equiv \{P_i(\mathbf{Z})\} = (\chi_1^2, \chi_2^2, W_1, W_2, \lambda)$.

·Expand $\mathbf{P}(\mathbf{Z} = \mathbf{Z}_0 + \mathbf{z}) = \mathbf{P}(\mathbf{Z}_0) + \mathbf{p}$ about $\mathbf{Z} = \mathbf{Z}_0$. One has (writing in component-form, with summation over repeated indices assumed)

$$p_i(\mathbf{Z}_0 + \mathbf{z}) = G_{ij}(\mathbf{Z}_0) z_j + \frac{1}{2} H_{ijk}(\mathbf{Z}_0) z_j z_k + (h.o.), \quad (1)$$

with $h.o. \equiv$ higher–order terms. For small enough $\mathbf{z}$, one has matrix equation

$$\mathbf{p} = \mathbf{G}_0 \cdot \mathbf{z}, \quad (2)$$

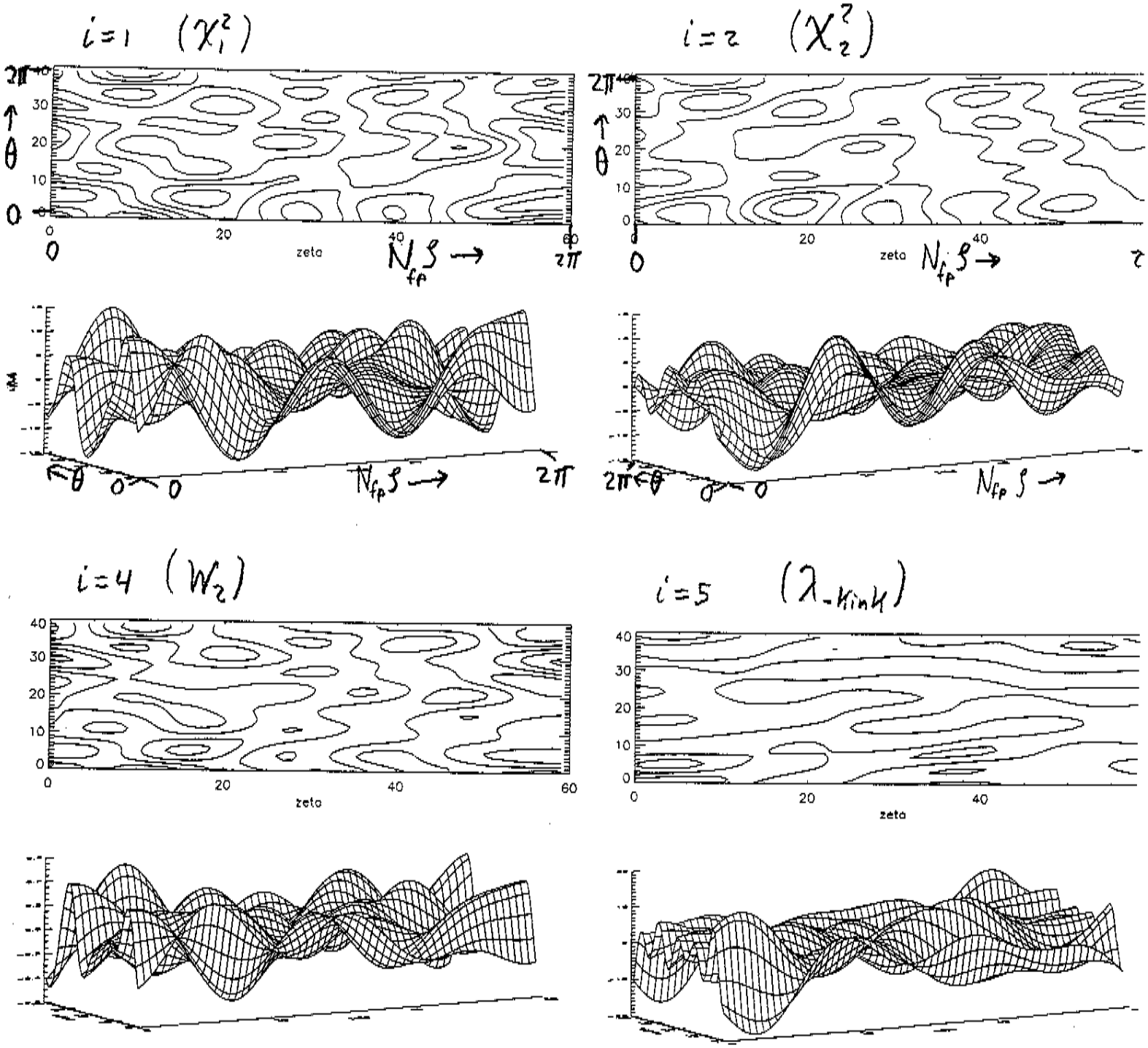
with $\mathbf{G}_0 \equiv \mathbf{G}(\mathbf{Z}_0)$ the $M \times N$ 'control matrix' at design point $\mathbf{Z}_0$. It may be inverted, using the SVD theorem
$\mathbf{G}_{M\times N} = \mathbf{U}_{M\times N} \cdot \mathbf{W}_{N\times N} \cdot \mathbf{V}^T_{N\times N}$,
with $\mathbf{U}, \mathbf{V}$ unitary matrices, and $\mathbf{W}$ a diagonal matrix.

·Taking the particular basis set $\boldsymbol{\pi}^{i=1,M}$ which have 1 in the ith position and 0 elsewhere, one has the corresponding set $\boldsymbol{\xi}^i$ of displacements

$$\boldsymbol{\xi}^i \equiv \mathbf{G}_0^{-1} \cdot \boldsymbol{\pi}^i,$$

physically representing a set of displacements which vary a single physics parameter P_i, leaving the others unchanged. These span the 'range' of $\mathbf{G}$. The $(N - M)$ vectors spanning the nullspace of $\mathbf{G}$ (change the configuration without modifying any of the P_i) are also important. To begin, we look at the $\boldsymbol{\xi}^i$.

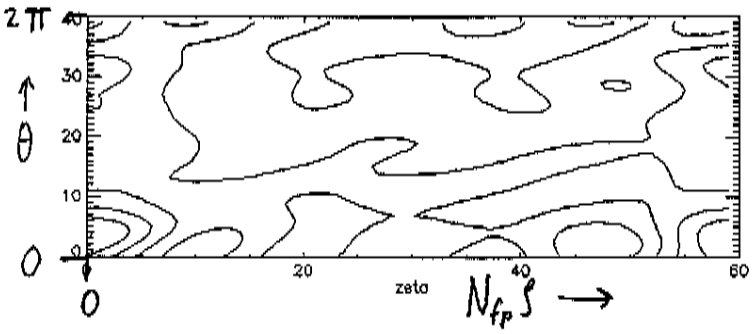
∘ c10 : Initial results, for $\mathbf{Z} = \mathbf{X}$, $N_z = N_x$:
(A)'More–constrained' case ($M_p = 5$).
·Plot $\xi^i(\theta, \zeta) \equiv \xi^i_R + \xi^i_z$ to assess the $\boldsymbol{\xi}^i$:



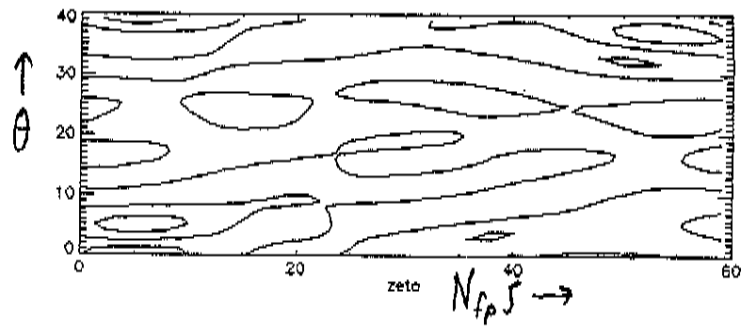
(B) 'Less-constrained' case ($M_p = 2$):

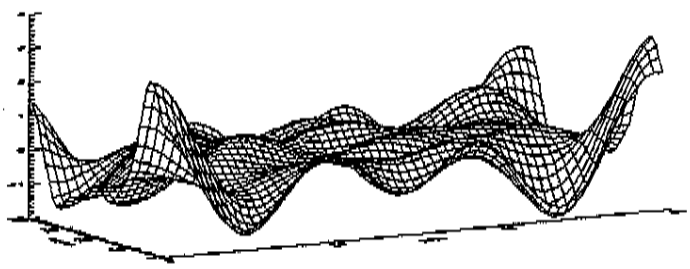
- Plot $\xi^{(i)}(\theta, \zeta)$:

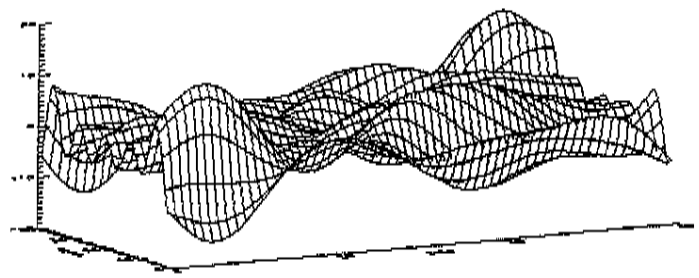
$i=1$ (χ_1^2)



$i=2$ (λ_{-kink})

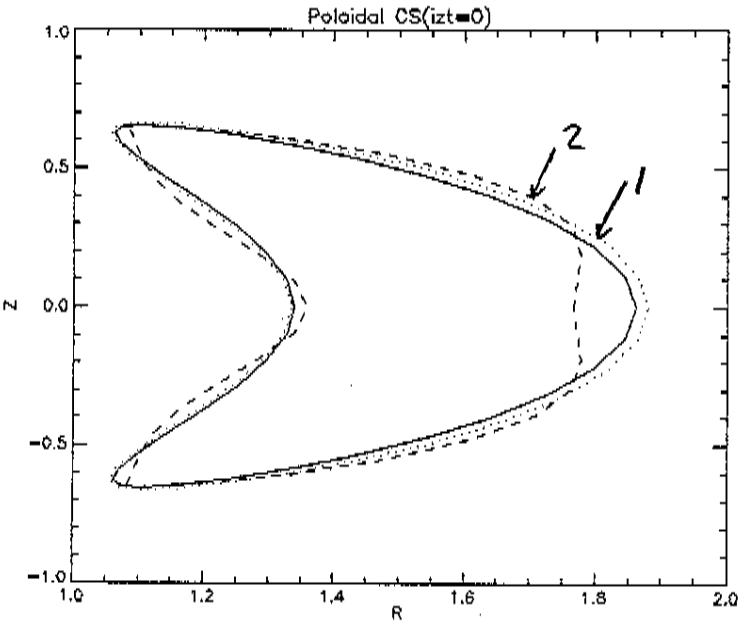




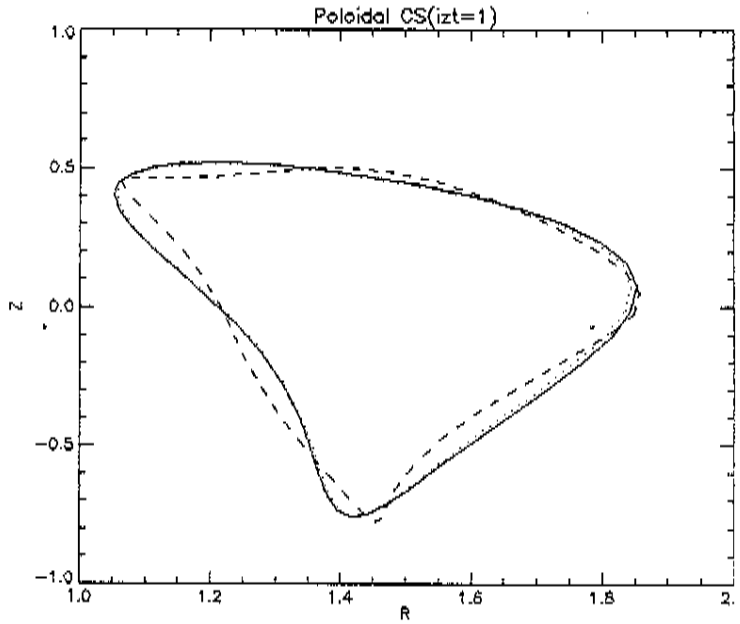


(B), cont:

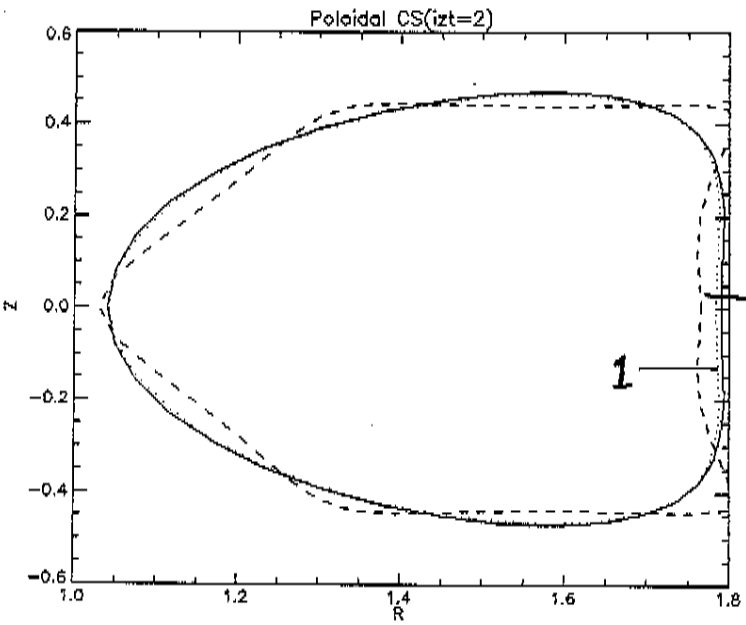
$N_{fp}\zeta = 0$:



$N_{fp}\zeta = \pi/2$:



$N_{fp}\zeta = \pi$:



(note indentedness)

i

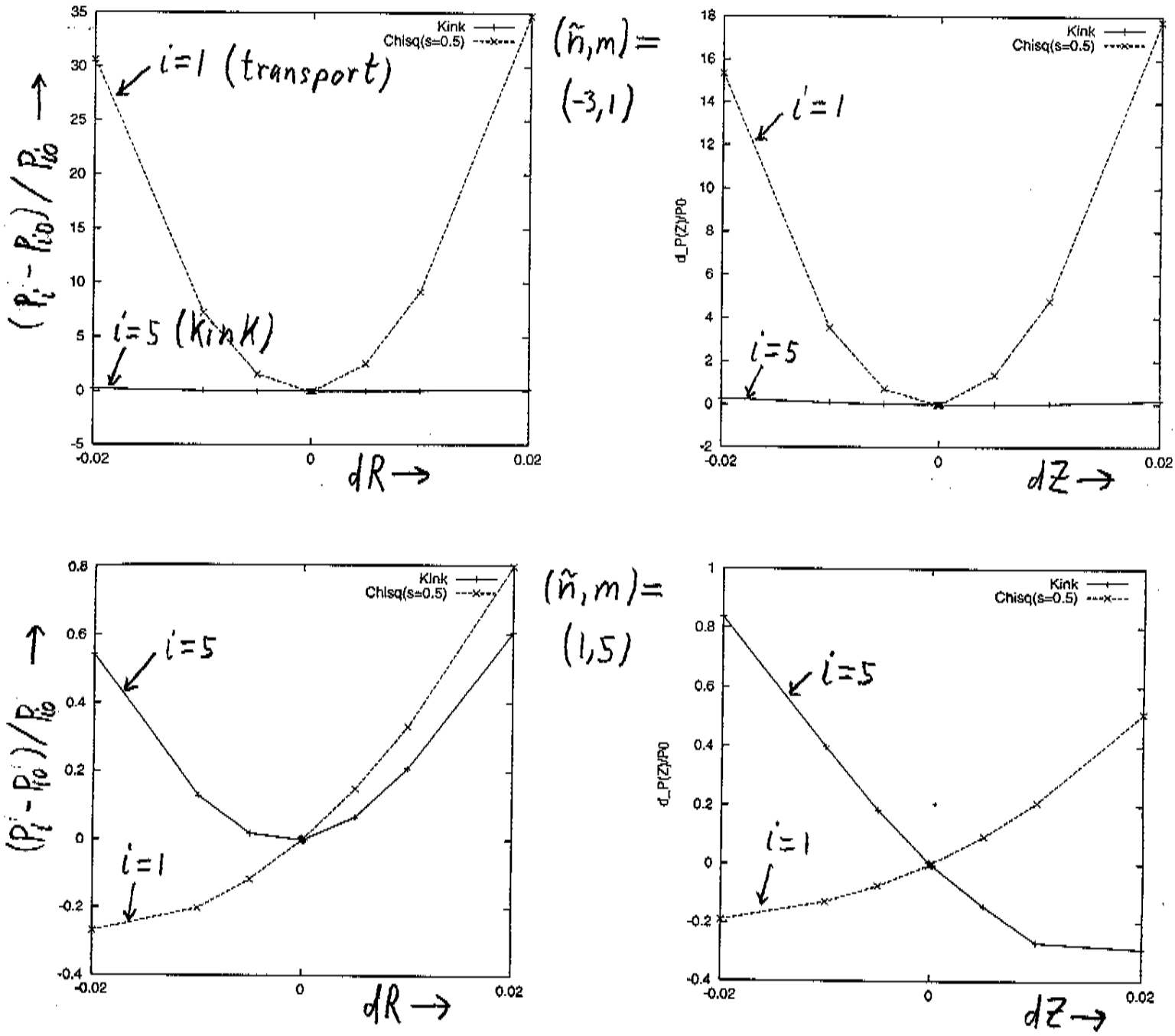
unpert = 0 ——— (c10)

1 .....

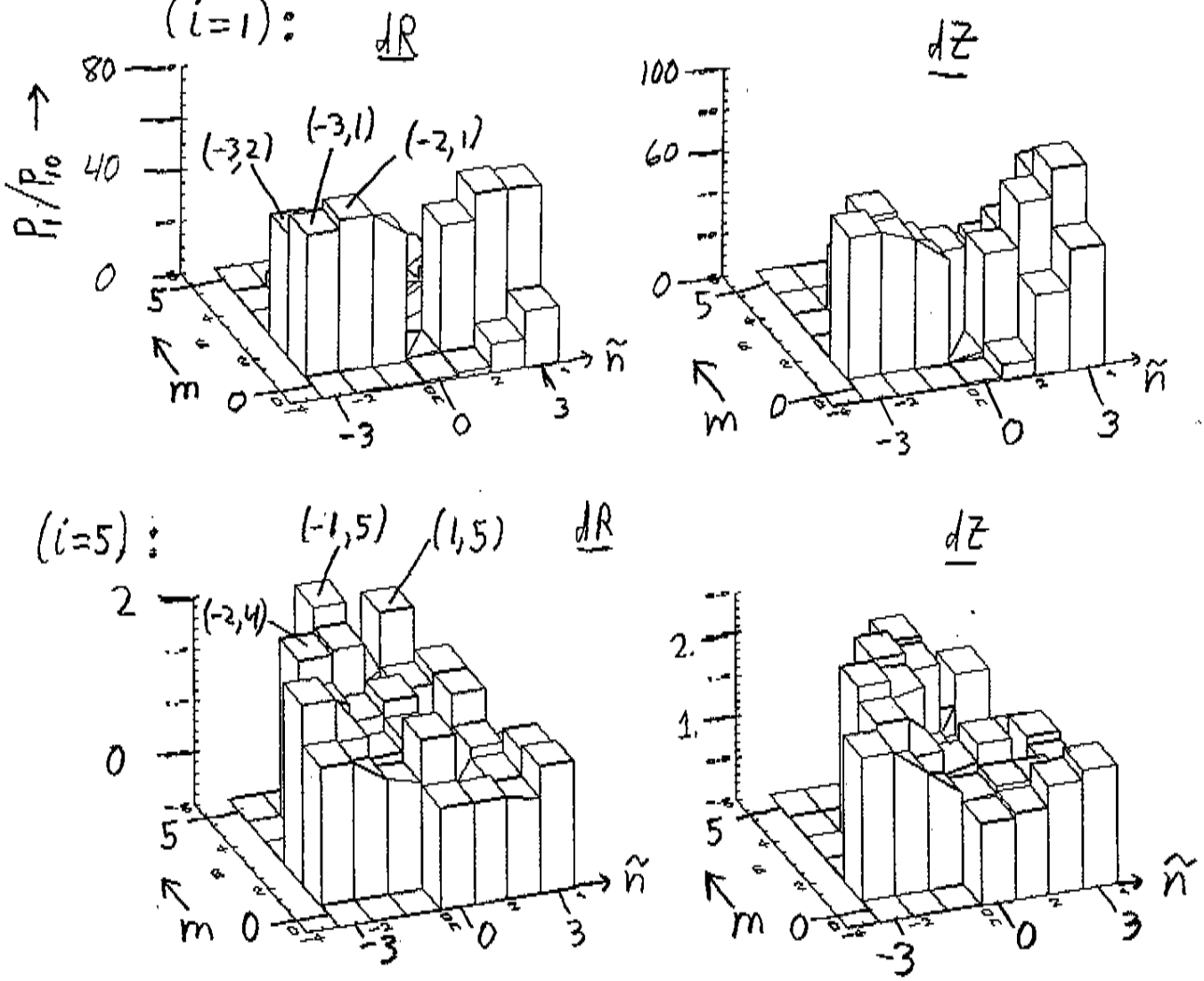
2 — — — —

∘Topography of **Z**-space:

The validity of Eqs. (1) or (2) depends on the typical scales of variation in **X**- or **Z**-space of the P_i. We have assessed this variation for all 78 X_j for the P_i presently being used, in the vicinity of the $\mathbf{X}_0$ = c10–c82 family of configurations. Some typical results:



○Histograms of sensitivity over $(\tilde{n}, m)$–plane:



∘Trends:

·As for the $\boldsymbol{\xi}^i$, the dominant $\mathbf{m}$ are similar for $i = 1-4$, and different from those for $i = 5$. Because of the symmetrizing action of the optimizer in creating c10, for almost all $\mathbf{m}$, the fractional changes p_i/p_{0i} for transport $(i = 1, 4)$ are much larger than that $(i = 5)$ for kink stability.

·The linear approximation Eq.(2) is valid only for fairly small displacements $\mathbf{z}$ (~ 1 millimeter) for some Z_j. However, the quadratic form (1) appears a *good* approximation for perturbations $\mathbf{z}$ on the order of a few centimeters. The simple $\mathbf{Z}$-space topography in the vicinity of c10 suggests that we can obtain a highly tractable model for study in this region. *E.g.*, from Eq.(1) one can compute the control matrix for *any* $\mathbf{Z}$ in this region:

$$G_{ij}(\mathbf{Z}) \equiv \partial p_i(\mathbf{Z})/\partial z_j = G_{ij}(\mathbf{Z}_0) + H_{ijk} z_k, \qquad (3)$$

and from this, find the correct $\boldsymbol{\xi}^i$ at any $\mathbf{Z}$, along with the extrema of the P_i, *etc.*.

⇒Here, we will present a 1st calculation of both G_{ij} *and* H_{ijk} around the c10 design point, within a 'reduced model' having $N_z \ll N_x$, and demonstrate numerically the 'proof of principle' of the CM method for this model.

∘Reducing the dimensionality of **Z**:

∘The dimensionality $N_x = N_z$ of the search space used up to now and $N_x - M_p$ of its null subspace are large (78 and 73, resp.). The smaller we can make these, the better. Here, we shall reduce N_z by 2 general methods:

(a)Removing the redundancy in the **X**-specification. This reduces N_z from $N_x = 78$ to $N_x/2 = 39$.

(b)Taking only the perturbations most effective in varying some P_i. Here, we choose the 4 most effective for P_1, and for P_5, thus further reducing N_z from 39 to 8.

∘Defining normal displacements to the plasma boundary:

For a plasma boundary defined by

$$\vec{X}(\theta,\phi) = R(\theta,\phi)\hat{R}(\phi) + Z(\theta,\phi)\hat{Z}, \tag{3}$$

a general displacement is $\vec{Y} = \delta\vec{X} = \delta R\hat{R} + \delta Z\hat{Z}$, and a normal displacement is

$$\vec{Y}\cdot\frac{\partial\vec{X}}{\partial\theta}\times\frac{\partial\vec{X}}{\partial\phi} = R\Big(\frac{\partial R}{\partial\theta}\delta Z - \frac{\partial Z}{\partial\theta}\delta R\Big). \tag{4}$$

Multiplying by $\cos(m\theta + n\phi)$ and integrating over θ and ϕ yields a matrix equation in the form

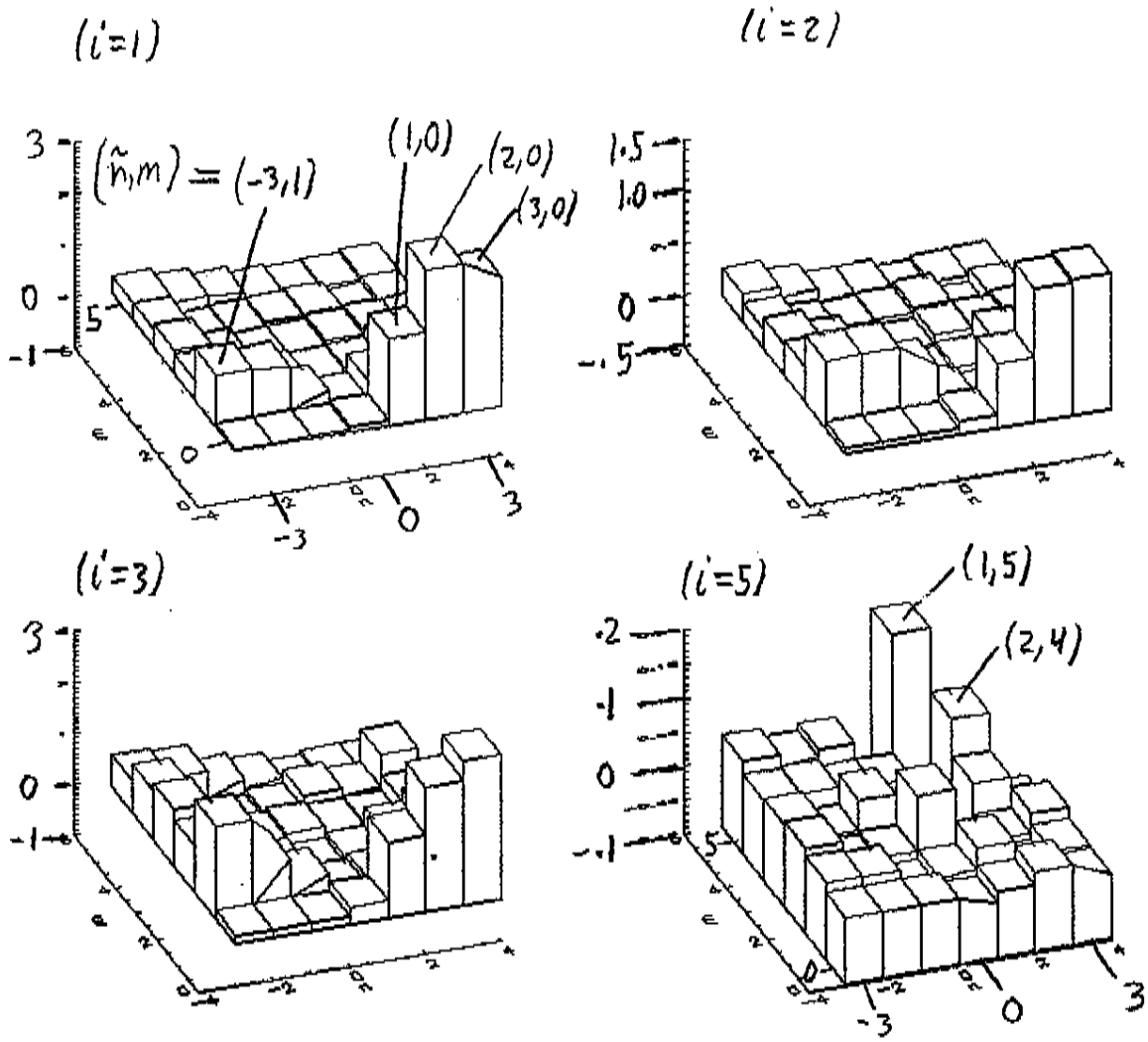
$$Y_i = \sum_j B_{ij}X_j. \tag{5}$$

Here $X_j (j = 1, \cdots, N)$ is the set of Fourier expansion coefficients of <u>both</u> δR and δZ, $Y_i (i = 1, \cdots, M)$ is the set of Fourier coefficients of the normal displacement to the plasma boundary, and B_{ij} is the $M \times N$ rectangular influence matrix that relates the two. Our goal is to calculate the X_j since these are required by VMEC.

Again, can do SVD-inversion of nonsquare matrix B to go from $\mathbf{Y} \equiv \{Y_{j=1,..N_y}\} \rightarrow \mathbf{X}$.

∘Reduced Model & 1st Calculation of G_{ij}, H_{ijk}:

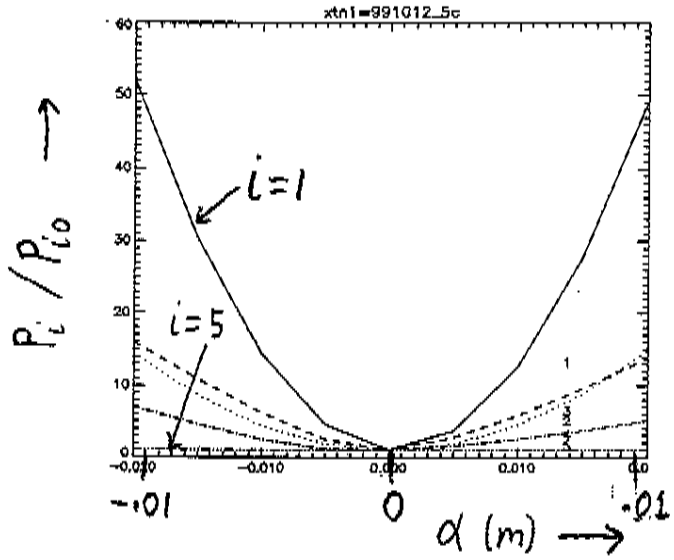
·Sensitivity histograms in the $N_z = 39$ **Z**-space:



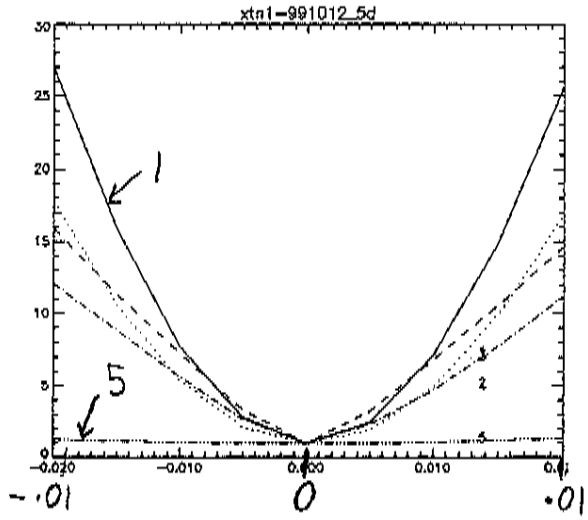
·Ranking $|P_i/P_{i0} - 1|$, select 4 most effective harmonics **m** for each of $i = 1, 5$:
·For P_1: $(\tilde{n}, m) = \{(1,0), (2,0), (3,0), (-3,1)\}$
·For P_5: $(\tilde{n}, m) = \{(1,3), (1,4), (2,4), (1,5)\}$.

○Topography of reduced space:

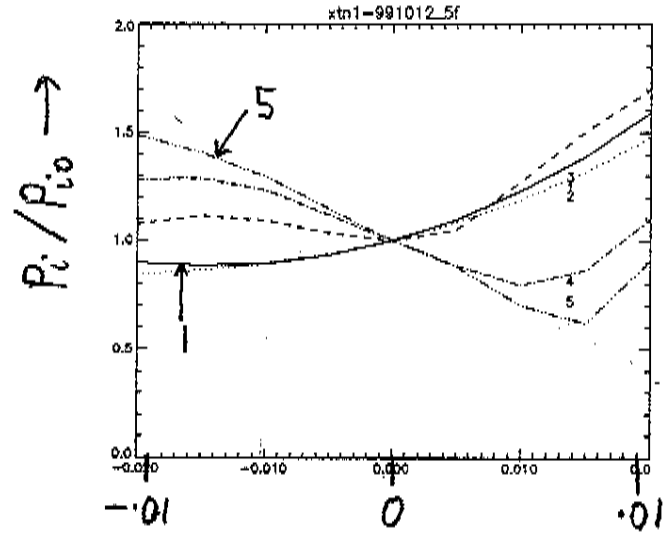
$j = 1,\ \underset{\sim}{m}_j = (1,0)$



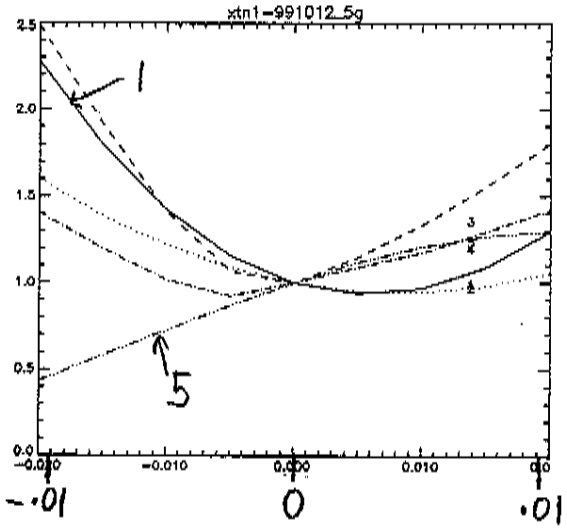
$j = 4,\ \underset{\sim}{m}_j = (-3,1)$



$j = 5,\ \underset{\sim}{m}_j = (1,3)$

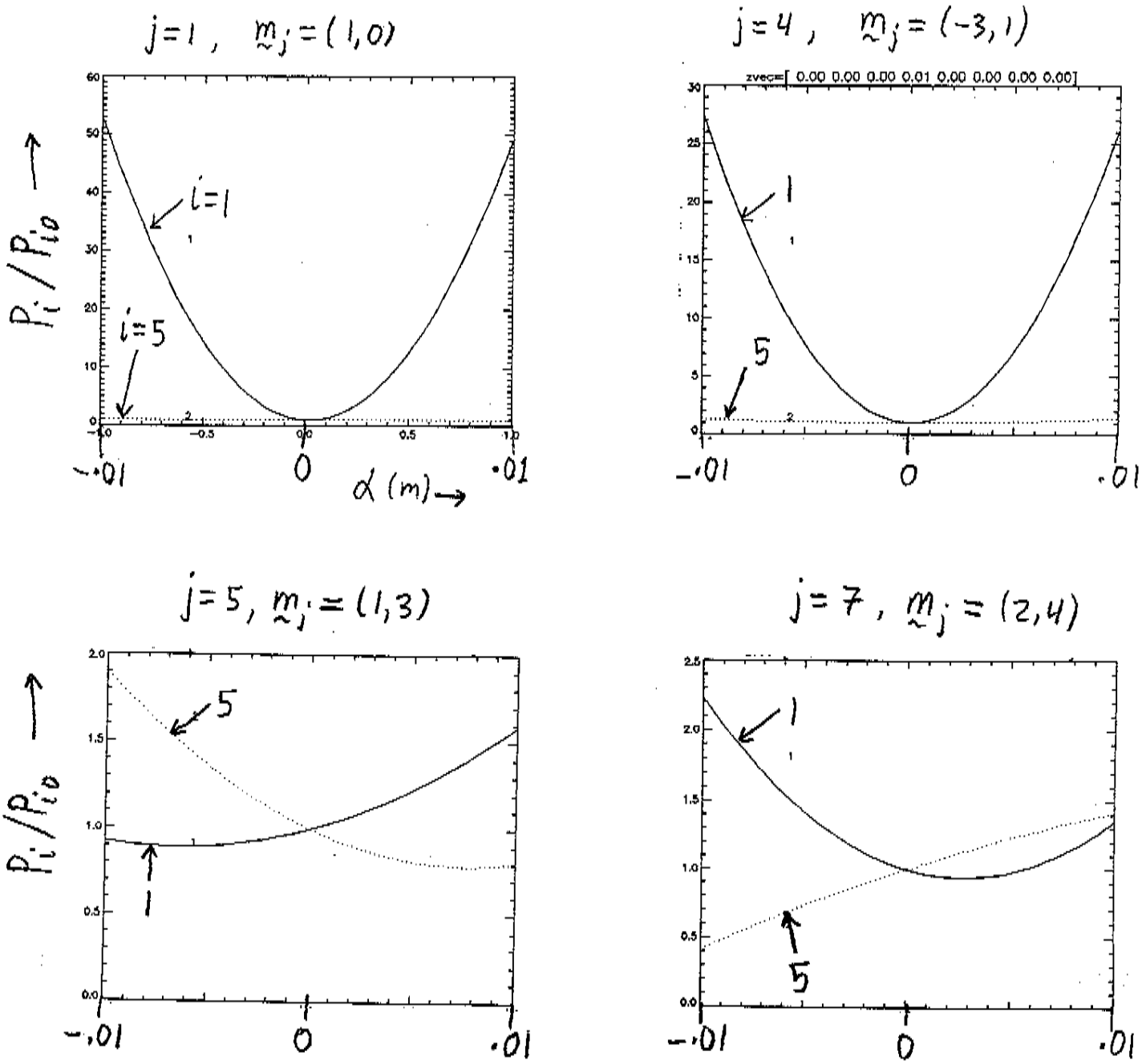


$j = 7,\ \underset{\sim}{m}_j = (2,4)$



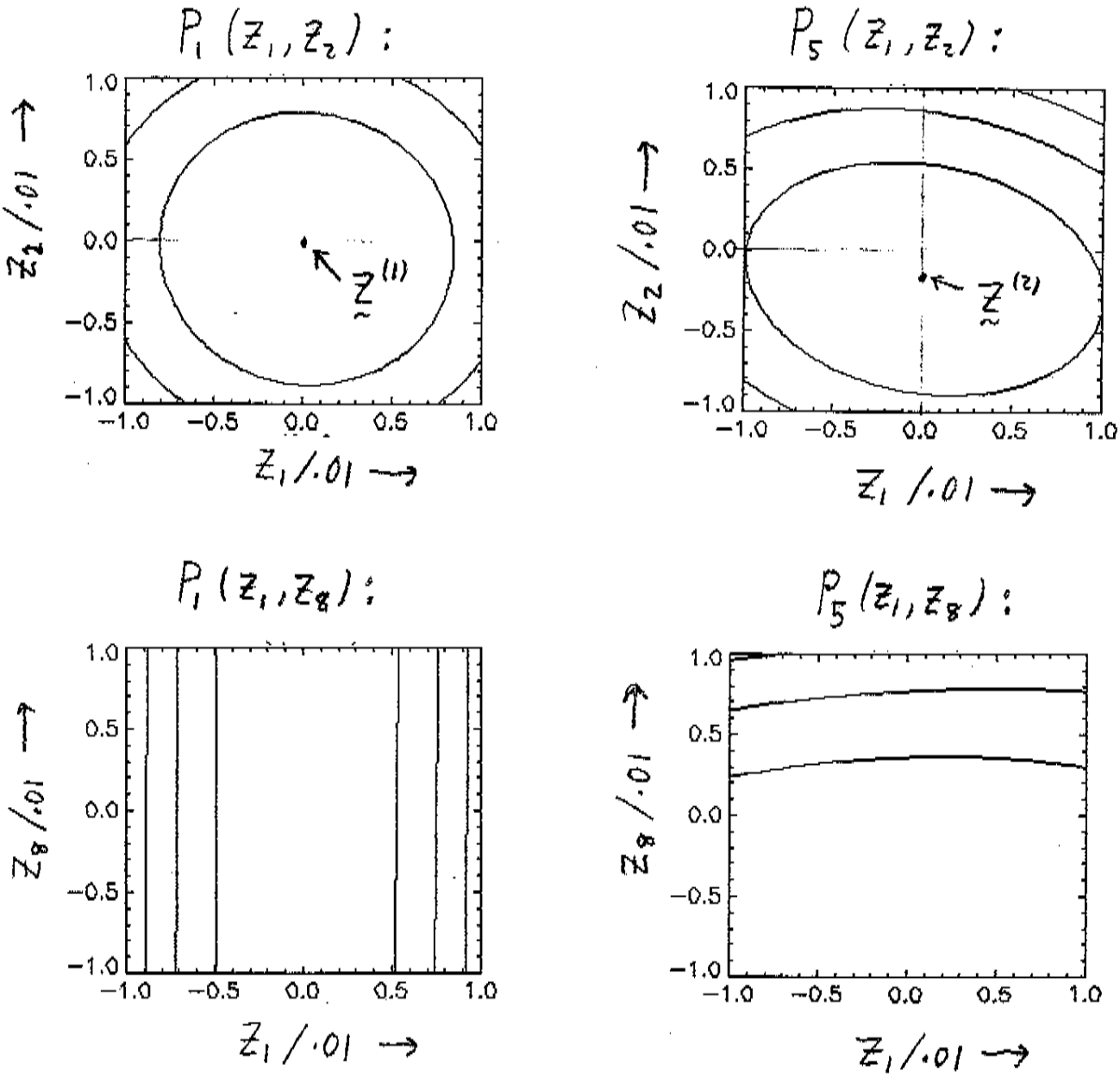
∘Distill G_{ij}, H_{ijk}⇒'Quadratic Model' for $N_z = 8$: (Requires $2N_z^2 = 128$ perturbed equilibria about $\mathbf{Z}_0$.)

·Compare this semi-analytic model with numerical results just shown:



∘Use Quadratic Model to analyse **Z**-space structure, compute $\boldsymbol{\xi}^{i}$'s, *etc.*.

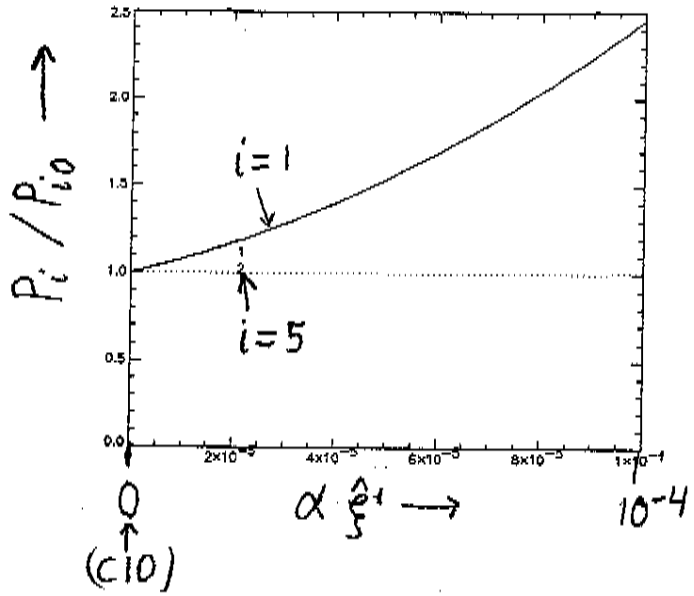
∘Plot $P_{1,5}$ versus pairs (z_{j_1}, z_{j_2}):

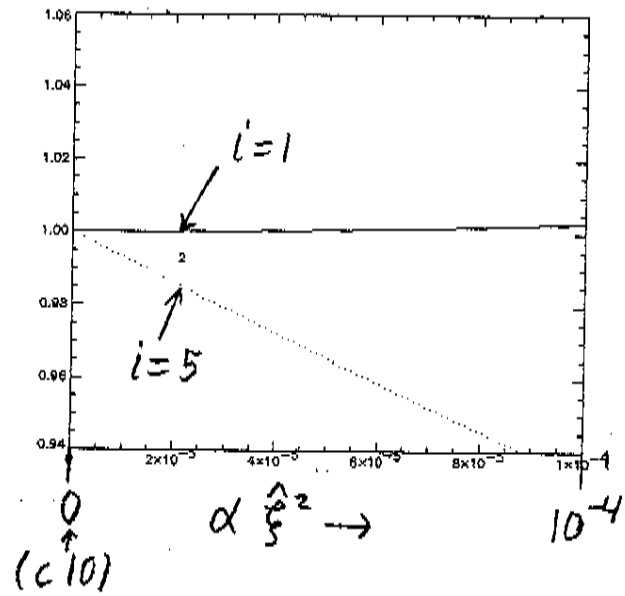


•(Note : z_1, z_8 almost perform job of ξ^1, ξ^5, viz, varying P_1, P_5 independently.)

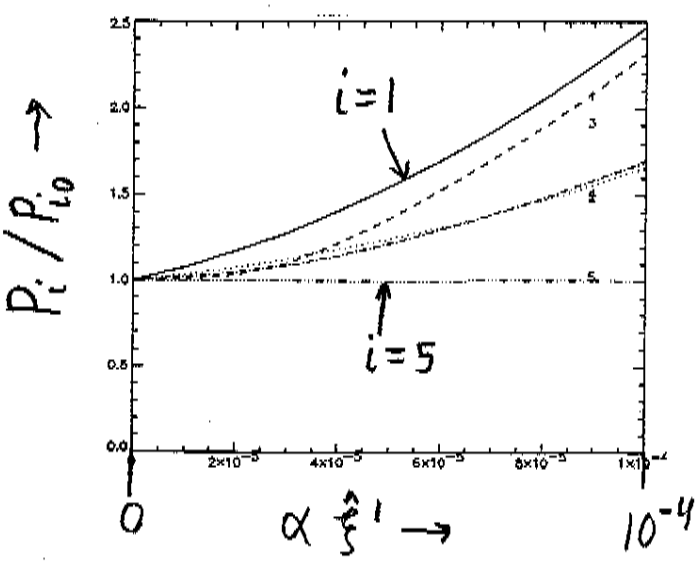
○'Proof of Principle' of CM method:

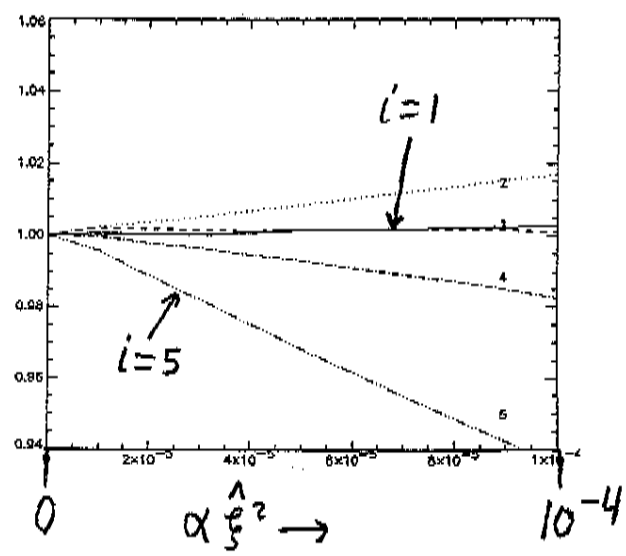
○Plot P_i versus $\alpha\xi^{j=1,5}$, verifying their independent control of $P_{1,5}$:





○Compare with numerical results from perturbed equilibria:



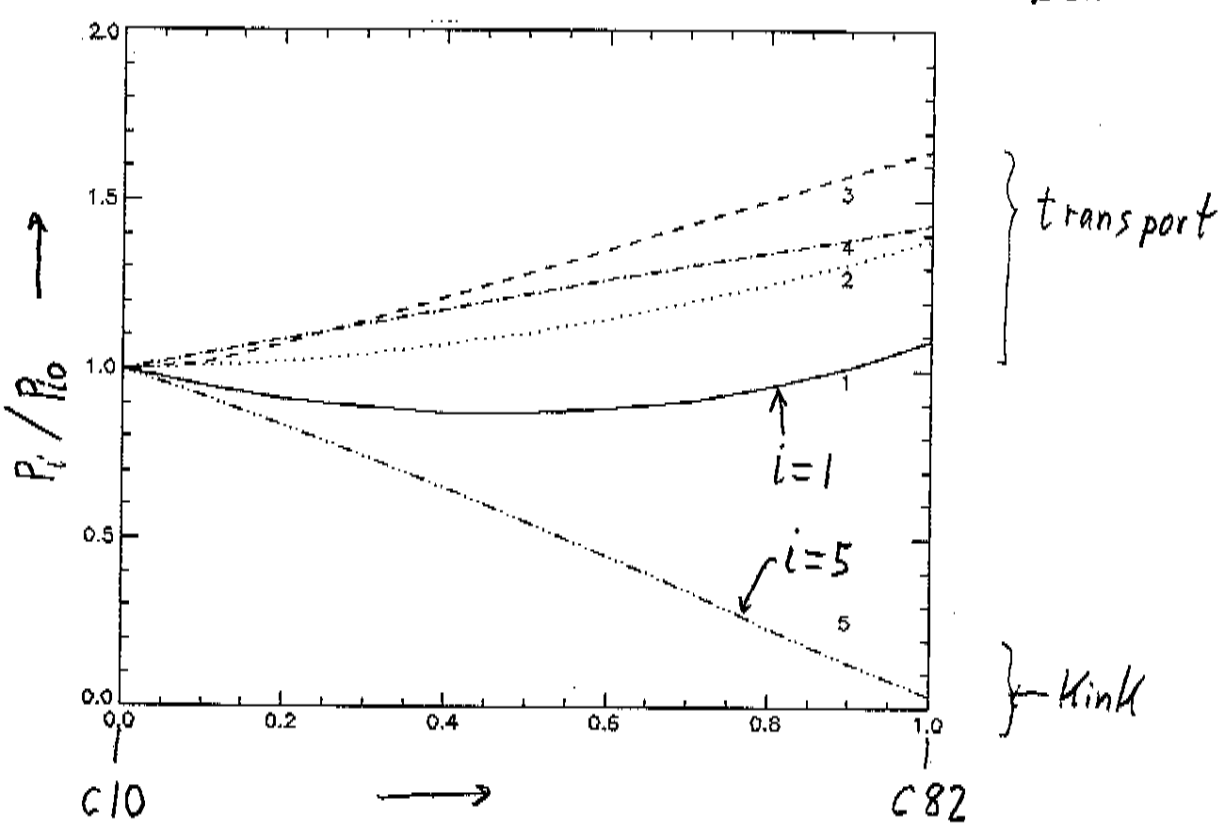


∘Other QAS Design Points:

·c10 was arrived at along an involved path of human interaction with the optimizer, and it is unclear that other regions of **Z**-space, which would have been reached from different starting points, might not yield superior configurations. Thus, we are starting to study other proposed QAS configurations[2,3] with the same methods, and to consider the variation of the P_i as one moves from one such point $\mathbf{Z}_0$ to another.

∘Warmup (**Z**'s nearby): The path c10 → c82:

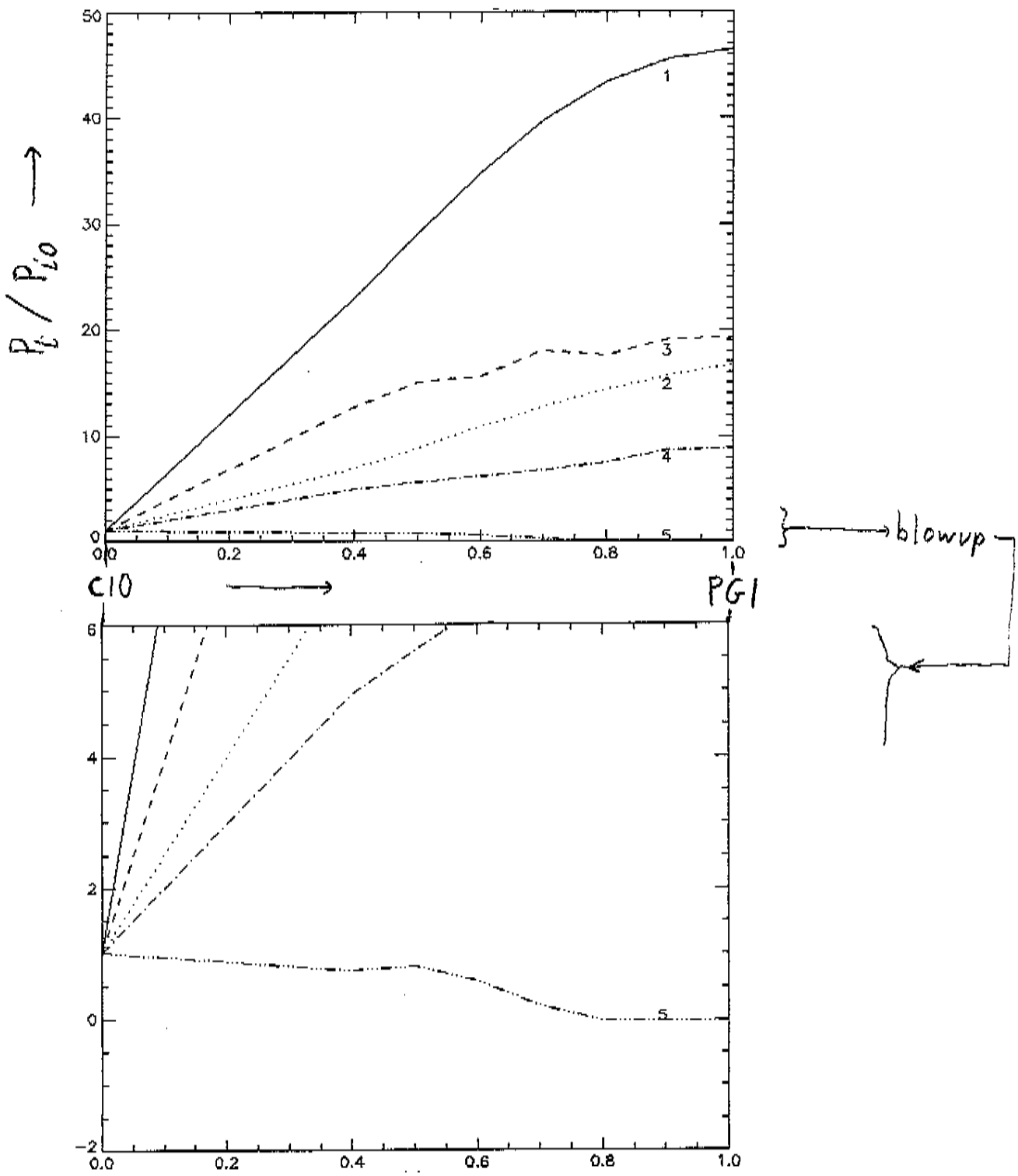
·c82 was obtained from c10 in an effort to stabilize the kink. The level of QA-ness was slightly degraded in compensation. This borne out by the P_i's along a straight-line path in **Z**-space: $|\underset{\sim}{X}_{c82} - \underset{\sim}{X}_{c10}| \simeq .041$ m



○The path c10 → PG1 (preliminary):

·Define norm $|\underset{\sim}{X}| \equiv [\sum_j X_j^2]^{1/2}$, $|\underset{\sim}{X}_{PG1} - \underset{\sim}{X}_{c10}| \simeq .228$ m

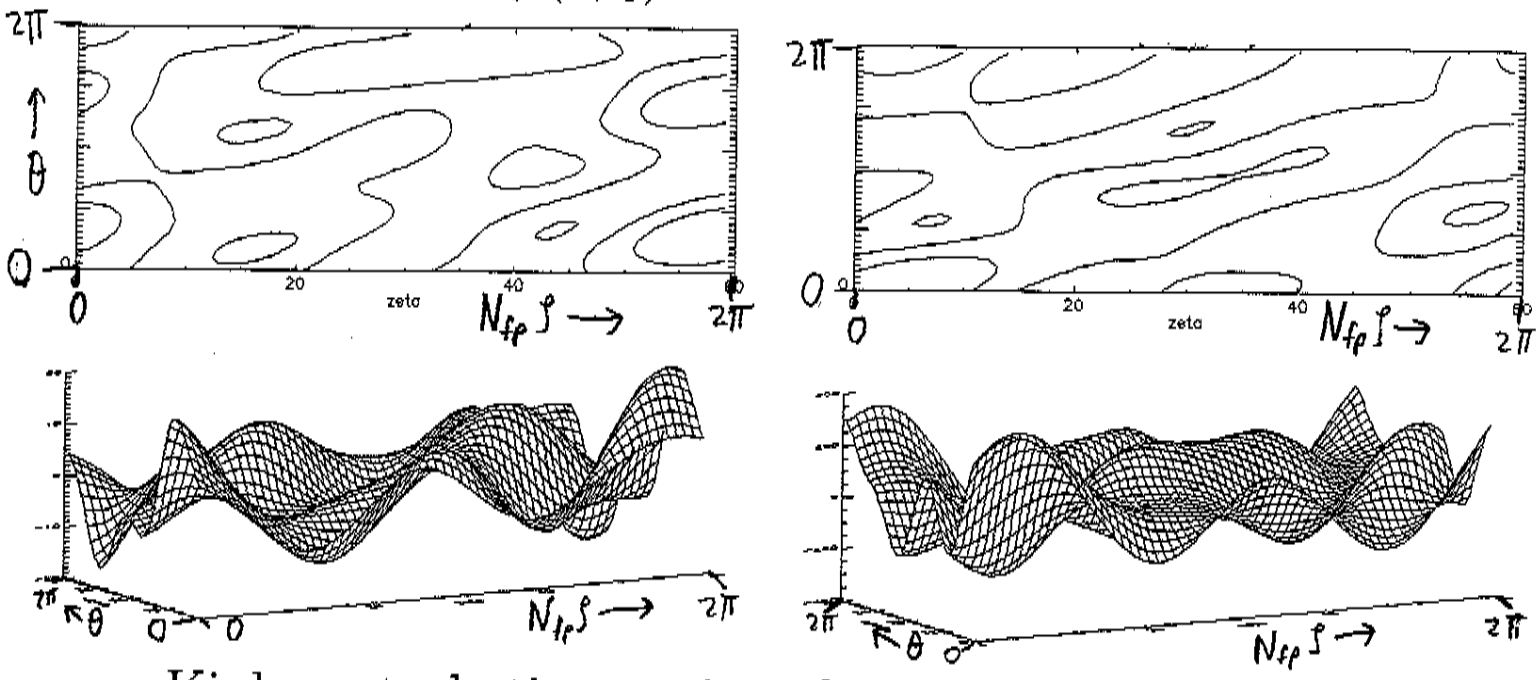
·PG1 characterized by much better kink stability, substantially worse QA-ness (mainly due to large mirror field $B_{m=0,\tilde{n}=1}$ present to enhance stability.



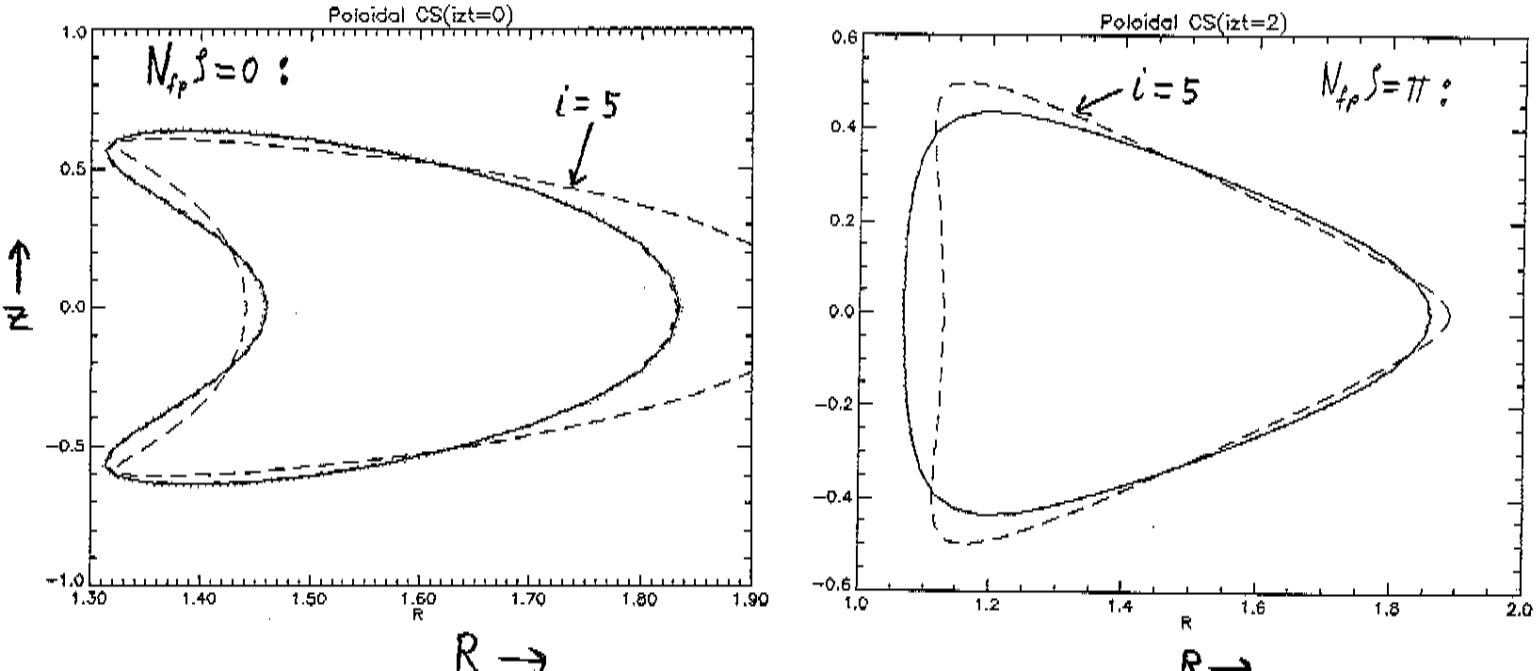
∘PG1: preliminary results:

·For $M_p = 5$:

·Not as much difference discernable between the transport and kink $\xi^i(\theta, \zeta)$:



·Kink perturbation again enhances the triangularity at $N_{fp}\zeta = \pi$, which is *positive* for PG1, consistent with tokamak-based intuition:



∘Better understanding and focussing the optimizer:

∘With the CM machinery now developed, we are in a position to look more closely at what the optimizer is doing, and how its performance might be improved.

∘Our examinations of the $P_i(\mathbf{Z})$ so far show some indications that the optimizer is doing what it should. E.g., c10 seems quite near an optimum in QA-ness.

∘However, some anomalies exist, *e.g.*, an apparent discrepancy between our quadratic-approximation and recent experiments Long-Poe has done near c10 with the optimizer.

∘Moreover, increased insight from our $\mathbf{Z}$-space topography explorations, and from better understanding of why a given perturbation has a given physics effect, should allow us to prune the space the optimizer needs to work in.

∘Application to coil design, flexibility & startup:

∘Instead of the specialization $\mathbf{Z} \to \mathbf{X}$ or $\mathbf{Y}$ considered up to now, specialize $\mathbf{Z} \to \mathbf{I}$, with $\mathbf{I} = \{I_{j=1,..N_z}\}$ describing the external currents: One may choose
(a)$I_i \to K_{\mathbf{m}_i} \equiv$ the Fourier amplitudes of the current potential $K(\theta, \zeta)$ (for coil design), or
(b)$I_i \to J_i \equiv$ the amount of current in the ith coil of a given coil set (for operational flexibility).
∘Then the $P_i(\mathbf{Z})$ can be computed almost as now, but using free-boundary VMEC.

∘Sometimes useful to extend the Z_j, P_i:
·Include $\langle\beta\rangle$ in the Z_j, (and maybe a parameter characterizing the peakedness of the pressure profile), and
·Extend the P_i to include P_6, = current–sheet complexity measure like that which Steve H cooked up last year.

∘Then applying the CM method just as earlier:
·$\boldsymbol{\xi}^6$ gives current–sheet changes which reduce the coil complexity, while keeping the physics performance unchanged.
·Including $\langle\beta\rangle$, the relative size of components of $\langle\beta\rangle$ to the other Z_j tell what changes need to be made as $\langle\beta\rangle$ is raised to maintain the physics.

References

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[3]N. Nakajima, M. Yokoyama, M. Okamoto, J. Nührenberg, *Plasma Physics Reports* **23**, 460 (1997).

[4]P. Garabedian, L.P. Ku, *Phys. Plasmas* **6**, 645 (1999).