

Connection to Quality Matrix Approach

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Addition to Harry's presentation: "Update on Control Matrix Studies".

PRESENT NESCOIL

Seeks a solution to the problem:

$$\text{Minimize } \|\mathbf{L}I - b\|_2. \quad (1)$$

Solution is by SVD: $\mathbf{L} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$, where matrices

\mathbf{L} and \mathbf{U} are $N_b \times N_I$, while $\mathbf{\Sigma}$ and \mathbf{V} are $N_I \times N_I$.

$\mathbf{\Sigma} = \text{diag}\{\sigma_j\}$ is a diagonal matrix of “singular values”.

NESCOIL presently **assumes** the “solution”:

$$\begin{aligned} \hat{I} = \mathbf{L}^+b \quad \text{where} \quad \mathbf{L}^+ = \mathbf{V}\mathbf{\Sigma}^+\mathbf{U}^T, \\ \mathbf{\Sigma}^+ = \text{diag}\{1/\sigma_j\}, \\ \text{and } 1/\sigma_j = 0 \quad \text{if } \sigma_j < \sigma_{cutoff} \approx 0. \end{aligned} \quad (2)$$

In the following, assume N_σ singular values satisfy $\sigma_j > \sigma_{cutoff}$.

The prescription defined in Eq. (2) for solving Eq. (1) is not unique. We can use this non-uniqueness to the advantage of coil design.

IMPROVED NESCOIL

Allen Boozer proposes

$$I = \hat{I} + \sum_{j=N_\sigma+1}^{N_I} c_j \vec{v}_j \quad (3)$$

as a preferred solution to Eq. (1). The \vec{v}_j are the last $N_I - N_\sigma$ column vectors of the orthogonal matrix \mathbf{V} . Each \vec{v}_j naturally satisfies

$$\mathbf{L}\vec{v}_j = 0. \quad (4)$$

On account of Eq. (4), the “solution” expressed by Eq. (3) using any chosen $\{c_j\}$ has the same residual (i.e., error) as the original NESCOIL solution:

$$\mathbf{L}I - b = \mathbf{L}\hat{I} - b. \quad (5)$$

Different $\{c_j\}$ will produce different contours of the current potential. We can choose the particular set, $\{c_j\}$, that minimizes some engineering criteria such as coil complexity and current density. If the residual is set at a level such that the original NESCOIL current sheet reconstructs the target plasma, the modified NESCOIL will

also reconstruct. By construction, however, the coil set will be “improved”.

Actually, Allen proposes to

$$\text{Minimize } \|\mathbf{W}(\mathbf{L}I - b)\|_2, \quad (6)$$

where \mathbf{W} is a weight matrix which is chosen to emphasize those pieces of the cost function that are responsible for the physics we really care about (quasisymmetry and stability), and to de-emphasize the remainder. The relation of \mathbf{W} to the so-called “Quality Matrix” is $\mathbf{Q} = \mathbf{W}^T \mathbf{W}$.

The solution of Eq. (6) depends on the SVD analysis of $\mathbf{W}\mathbf{L}$. Suppose, for a moment, that \mathbf{W} is a diagonal matrix and a small number of weights, W_{ii} , are large to emphasize a few of the N_b equations:

$$W_{ii}(L_{ij}I_j - b_i) = 0. \quad (7)$$

It is easy to see that the number of significant singular values (rank) of $\mathbf{W}\mathbf{L}$ will equal the number of large weights. The fewer the number of large weights, the smaller the rank of $\mathbf{W}\mathbf{L}$ and the larger the associated null space.

The actual weight matrix we should use in Eq. (6) is not diagonal. It is essentially the (square root of the) Hessian calculated by our Control Matrix procedure: Recall, from Harry, that

$$p_i = G_{ij}z_j + \frac{1}{2}H_{ijk}z_jz_k \quad (8)$$

describes the variation of selected physics parameters as a target configuration shape is varied. Near an optimum physics configuration, the Hessian tells us how rapidly the physics varies away from the optimum. To convert the Hessian into a quality matrix, we must first transform from z -space to b -space using

$$z_j = T b_j, \quad (9)$$

to obtain

$$p_i = \tilde{G}_{ij}b_j + \frac{1}{2}\tilde{H}_{ijk}b_jb_k. \quad (10)$$

The Quality Matrix is obtained by a contraction of the Hessian tensor, by defining the overall quality of the configuration as a particular weighted combination of the

physics parameters:

$$\text{quality} = \sum_i \lambda_i p_i, \quad (11)$$

just as the optimizer does. The QM is

$$Q_{jk} = \sum_i \frac{\lambda_i}{2} \tilde{H}_{ijk}. \quad (12)$$

The “square root” of \mathbf{Q} gives the desired weight matrix \mathbf{W} . We can expect that the rank of \mathbf{W} is less than N_b implying more wiggle room for improving the coils.