NESCOIL Improvement by Use of Null Space Solutions -Related to Control Matrix and Quality Matrix Approaches

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A follow-on to Harry's 12/2/99 presentation: "Update on Control Matrix Studies".

PRESENT NESCOIL

Seeks a solution to the problem:

$$\text{Minimize} \quad \|\mathbf{L}I - b\|_2. \tag{1}$$

Solution is by SVD: $\mathbf{L} = \mathbf{U\Sigma V^T}$, where matrices $\mathbf{L}$ and $\mathbf{U}$ are $N_b \times N_I$, while $\mathbf{\Sigma}$ and $\mathbf{V}$ are $N_I \times N_I$.

$\mathbf{\Sigma} = \text{diag}\{\sigma_j\}$ is a diagonal matrix of "singular values".

NESCOIL presently **assumes** the "solution":

$$\hat{I} = \mathbf{L}^+ b \quad \text{where} \quad \mathbf{L}^+ = \mathbf{V\Sigma^+U^T},$$
$$\mathbf{\Sigma}^+ = \text{diag}\,\{1/\sigma_j\},$$
$$\text{and} \quad 1/\sigma_j = 0 \quad \text{if} \quad \sigma_j < \sigma_{cutoff} \approx 0. \tag{2}$$

In the following, assume N_σ singular values satisfy $\sigma_j > \sigma_{cutoff}$.

The prescription defined in Eq. (2) for solving Eq. (1) is not unique. We can use this non-uniqueness to the advantage of coil design.

IMPROVED NESCOIL

Allen Boozer proposes

$$I = \hat{I} + \sum_{j=N_\sigma+1}^{N_I} c_j \vec{v}_j \tag{3}$$

as a preferred solution to Eq. (1). The $\vec{v}_j$ are the last $N_I - N_\sigma$ column vectors of the orthogonal matrix $\mathbf{V}$. Each $\vec{v}_j$ naturally satisfies

$$\mathbf{L}\vec{v}_j = 0. \tag{4}$$

On account of Eq. (4), the "solution" expressed by Eq. (3) using any chosen $\{c_j\}$ has the same residual (i.e., error) as the original NESCOIL solution:

$$\mathbf{L}I - b = \mathbf{L}\hat{I} - b. \tag{5}$$

Different $\{c_j\}$ will produce different contours of the current potential. We can choose the particular set, $\{c_j\}$, that minimizes some engineering criteria such as coil complexity and currrent density. If the residual is set at a level such that the original NESCOIL current sheet reconstructs the target plasma, the modified NESCOIL will also reconstruct. By construction, however, the coil set will be "improved".

Actually, Allen proposes to

$$\text{Minimize} \quad \|\mathbf{W}(\mathbf{L}I - b)\|_2, \tag{6}$$

where $\mathbf{W}$ is a weight matrix which is chosen to emphasize those pieces of the cost function that are responsible for the physics we really care about (quasisymmetry and stability), and to de-emphasize the remainder. The relation of $\mathbf{W}$ to the so-called "Quality Matrix" is $\mathbf{Q} = \mathbf{W^TW}$.

The solution of Eq. (6) depends on the SVD anaysis of $\mathbf{WL}$. Suppose, for a moment, that $\mathbf{W}$ is a diagonal matrix and a small number of weights, W_{ii}, are large to emphasize a few of the N_b equations:

$$W_{ii}(L_{ij}I_j - b_i) = 0. \tag{7}$$

It is easy to see that the number of significant singular values (rank) of $\mathbf{WL}$ will equal the number of large weights. The fewer the number of large weights, the smaller the rank of $\mathbf{WL}$ and the larger the associated null space.

The actual weight matrix we should use in Eq. (6) is not diagonal. It is essentially the (square root of the) Hessian calculated by our Control Matrix procedure: Recall, from Harry, that

$$p_i = G_{ij} z_j + \frac{1}{2} H_{ijk} z_j z_k \tag{8}$$

describes the variation of selected physics parameters as a target configuration shape is varied. Near an optimum physics configuration, the Hessian tells us how rapidly the physics varies away from the optimum. To convert the Hessian into a quality matrix, we must first transform from z-space to b-space using

$$z_j = T b_j, \tag{9}$$

to obtain

$$p_i = \tilde{G}_{ij} b_j + \frac{1}{2} \tilde{H}_{ijk} b_j b_k. \tag{10}$$

The Quality Matrix is obtained by a contraction of the Hessian tensor, by defining the overall quality of the configuration as a particular weighted combination of the physics parameters:

$$\text{Objective Function} = \sum_i \lambda_i p_i, \tag{11}$$

just as the optimizer does. The QM is

$$Q_{jk} = \sum_i \frac{\lambda_i}{2} \tilde{H}_{ijk}. \tag{12}$$

The "square root" of $\mathbf{Q}$ gives the desired weight matrix $\mathbf{W}$. We can expect that the rank of $\mathbf{W}$ is less than N_b implying more wiggle room for improving the coils.

Yet Another Approach

Just as we have seen that using the null space of the inductance matrix $\mathbf{L}$ should lead to improved coils for a given target plasma shape, we can exploit another null space to achieve a similar goal, namely the null space of the control matrix $\mathbf{G}$. Here, given a target shape, $\mathbf{Z_0}$ we construct null space vectors z_n of $\mathbf{G}$ such that

$$\mathbf{G z_n = 0}. \tag{13}$$

Thus, configurations with shapes defined by

$$\text{New Shape} \quad = \mathbf{Z_0} + \sum_{n=1}^{N_z - r} c_n \mathbf{z_n} \tag{14}$$

have the same physics as the target $\mathbf{Z_0}$. ($r = \text{Rank}(\mathbf{G})$). However, since the shapes for different $\{c_n\}$ are different, so too will be the NESCOIL current sheet solutions. As before, we can choose the $\{c_n\}$ to minimize sum engineering cost function, such as complexity and current density.

Square Root of a Matrix

Problem Statement: Given an $N \times N$ symmetric matrix, $\mathbf{Q}$, calculate an $N \times N$ symmetric matrix $\mathbf{W}$ such that $\mathbf{Q} = \mathbf{W^T W}$.

Solution: $\mathbf{W}$ is eigenmatrix of $\mathbf{Q}$

To see this, consider the basic equations of SVD analysis:

$$\mathbf{W} = \mathbf{U \Sigma V^T}. \tag{15}$$

Then

$$\begin{aligned} \mathbf{W W^T} &= (\mathbf{U \Sigma V^T})(\mathbf{V \Sigma^T U^T}), \\ \Rightarrow (\mathbf{W W^T})\mathbf{U} &= \mathbf{U}(\mathbf{\Sigma \Sigma^T}). \end{aligned} \tag{16}$$

Similarly

$$(\mathbf{W^T W})\mathbf{V} = \mathbf{V}(\mathbf{\Sigma^T \Sigma}). \tag{17}$$

Thus, the columns of $\mathbf{U}$ and $\mathbf{V}$ are eigenfunctions of $\mathbf{W W^T}$, and $\mathbf{W^T W}$, respectively.

Also, Eq. (15) $\Rightarrow$

$$\begin{aligned} \mathbf{W V} &= \mathbf{U \Sigma}, \\ \text{and} \quad \mathbf{W^T U} &= \mathbf{V \Sigma^T}. \end{aligned} \tag{18}$$

Now, for the "square root" problem, since $\mathbf{W}$ is symmetric, it follows that $\mathbf{U}$ and $\mathbf{V}$ are equal, and are simply the eigenfunctions of $\mathbf{W}^{\mathbf{T}}\mathbf{W}$, which is simply $\mathbf{Q}$. Then the desired solution, $\mathbf{W}$ is given by Eq. (15).