Evolution equation for stellarator equilibria

L. E. Zakharov

PPPL, Princeton NJ

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1. Notations

Flux coordinate system

$$r = r(a,\theta,\zeta), \quad z = z(a,\theta,\zeta), \quad \phi = \zeta(a,\theta,\zeta). \tag{1.1}$$

Metric tensor

$$\begin{aligned} dl^2 =& dr^2 + dz^2 + r^2 d\phi^2 \\ &\equiv g_{aa} da^2 + 2g_{a\theta} da d\theta + g_{\theta\theta} d\theta^2 + 2g_{a\zeta} da d\zeta + 2g_{\theta\zeta} d\theta d\zeta \\ &\quad + g_{\zeta\zeta} d\zeta^2, \\ g_{aa} =& r'_a r'_a + z'_a z'_a + r^2 \phi'_a \phi'_a, \quad g_{a\theta} = r'_a r'_\theta + z'_a z'_\theta + r^2 \phi'_a \phi'_\theta, \\ g_{\theta\theta} =& r'_\theta r'_\theta + z'_\theta z'_\theta + r^2 \phi'_\theta \phi'_\theta, \quad g_{a\zeta} = r'_a r'_\zeta + z'_a z'_\zeta + r^2 \phi'_a \phi'_\zeta, \\ g_{\theta\zeta} =& r'_\theta r'_\zeta + z'_\theta z'_\zeta + r^2 \phi'_\theta \phi'_\zeta, \quad g_{\zeta\zeta} = r'_\zeta r'_\zeta + z'_\zeta z'_\zeta + r^2 \phi'_\zeta \phi'_\zeta. \end{aligned} \tag{1.2}$$

Covariant representation of the vector potential

$$\mathbf{A} = -\,\eta\nabla a + \Phi\nabla\theta + \Psi\nabla\zeta, \tag{1.3}$$

where Φ, Ψ are the toroidal and poloidal fluxes. Contravariant representation of the magnetic field in terms of magnetic fluxes

$$\begin{aligned}
&\mathbf{B} = \nabla\times\mathbf{A} = B^\theta\nabla\zeta\times\nabla a + B^\zeta\nabla a\times\nabla\theta,\\
&\sqrt{g}B^a = \sqrt{g}\mathbf{B}\cdot\nabla a = 0,\\
&\sqrt{g}B^\theta = \sqrt{g}\mathbf{B}\cdot\nabla\theta = -\Psi'_a - \eta'_\zeta,\\
&\sqrt{g}B^\zeta = \sqrt{g}\mathbf{B}\cdot\nabla\zeta = \Phi'_a + \eta'_\theta.
\end{aligned} \tag{1.4}$$

Contravariant representation of the current density in terms of currents

$$\begin{aligned}
&\sqrt{g}j^a = 0,\\
&\sqrt{g}j^\theta = -F'(a) - \nu'_\zeta,\\
&\sqrt{g}j^\zeta = J'(a) + \nu'_\theta,
\end{aligned} \tag{1.5}$$

where F, J are the total poloidal and toroidal currents, correspondingly.

Covariant form of Ampere's law

$$\begin{aligned} \mu_0(F'+\nu'_\zeta) =& (B_\zeta)'_a-(B_a)'_\zeta, \quad 0=(B_\zeta)'_\theta-(B_\theta)'_\zeta, \\ \mu_0(J'+\nu'_\theta) =& (B_\theta)'_a-(B_a)'_\theta, \end{aligned} \tag{1.6}$$

is equivalent to following expressions for covariant components of the magnetic field

$$\begin{aligned} \mu_0(\nu+\phi'_a) &= B_a \equiv -\frac{g_{a\theta}}{\sqrt{g}}(\Psi'+\eta'_\zeta)+\frac{g_{a\zeta}}{\sqrt{g}}(\Phi'+\eta'_\theta), \\ \mu_0(J+\phi'_\theta) &= B_\theta \equiv -\frac{g_{\theta\theta}}{\sqrt{g}}(\Psi'+\eta'_\zeta)+\frac{g_{\theta\zeta}}{\sqrt{g}}(\Phi'+\eta'_\theta), \\ \mu_0(F+\phi'_\zeta) &= B_\zeta \equiv -\frac{g_{\zeta\theta}}{\sqrt{g}}(\Psi'+\eta'_\zeta)+\frac{g_{\zeta\zeta}}{\sqrt{g}}(\Phi'+\eta'_\theta). \end{aligned} \tag{1.7}$$

The pressure balance

$$\begin{aligned} &\mu_0 p'\sqrt{g} = -(\Phi'+\eta'_\theta)(F'+\nu'_\zeta)+(\Psi'+\eta'_\zeta)(J'+\nu'_\theta), , \\ &\mu_0 p'(\sqrt{g})_0 = -\Phi'F'+\Psi'J', \end{aligned} \tag{1.8}$$

where $(\ldots)_0$ designates averaging over angle variables θ, ζ.

2. Evolution equation

Covariant representation of the electric field

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi_E - \eta \nabla a + \Phi \nabla \theta + \Psi \nabla \zeta. \tag{2.1}$$

Averaged parallel component of the Ohm law $\sigma_\parallel(\mathbf{B}\cdot\mathbf{E}) = (\mathbf{B}\cdot\mathbf{j})$

$$\begin{aligned} (\sqrt{g}\mathbf{B}\cdot\mathbf{E})_0 =& \Psi'_a\Phi'_t - \Phi'_a\Psi'_t, \\ (\sqrt{g}\mathbf{B}\cdot\mathbf{j})_0 =& FJ' - JF' = FJ' - JF' \end{aligned} \tag{2.2}$$

gives the evolution equation in its general form (V. D. Shafranov, Reviews of Plasma Physics, v.2, 1962) as

$$\Psi'_a\Phi'_t - \Phi'_a\Psi'_t = \frac{1}{\sigma_\parallel}(FJ' - JF'). \tag{2.3}$$

In the form of a magnetic diffusion equation it can be rewritten as

$$\begin{aligned} \Psi'_a\Phi'_t - \Phi'_a\Psi'_t =& \frac{1}{\mu_0\sigma_\parallel}\left[(F+\mu J)\left(-\frac{g_{\theta\theta}}{\sqrt{g}}(\Psi' + \eta'_\zeta) + \frac{g_{\theta\zeta}}{\sqrt{g}}(\Phi' + \eta'_\theta)\right)'_{0a} \right. \\ & \left. + \frac{\mu_0 p'(\sqrt{g})_0 J}{\Phi'}\right]. \end{aligned} \tag{2.4}$$

In the flux coordinates with Φ as a radial coordinate (like in VMEC), it can be simplified as

$$
\begin{aligned}
\Psi_t' = & \frac{1}{\mu_0 \sigma_\parallel} \left[(F + \mu J) \left(\frac{g_{\theta\theta}}{\sqrt{g}} (\Psi_\Phi' + \eta_\zeta') - \frac{g_{\theta\zeta}}{\sqrt{g}} (1 + \eta_\theta') \right)_{0\Phi}' \right. \\
& \left. - \mu_0 p' (\sqrt{g})_0 J \right] .
\end{aligned} \tag{2.5}
$$

It can be also rewritten as the evolution equation for the rotational transform $\mu \equiv 1/q$

$$
\begin{aligned}
\mu_t' = & \left\{ \frac{1}{\mu_0 \sigma_\parallel} \left[(F + \mu J) \left(\frac{g_{\theta\theta}}{\sqrt{g}} (\mu + \eta_\zeta') - \frac{g_{\theta\zeta}}{\sqrt{g}} (1 + \eta_\theta') \right)_{0\Phi}' \right. \right. \\
& \left. \left. - 4\pi p' (\sqrt{g})_0 J \right] \right\}_\Phi'
\end{aligned} \tag{2.6}
$$

and, thus, can be used to produce μ as an input profile for VMEC.