On-Axis Transform In QuasiSymmetry

Ilon Joseph & Allen Boozer

Columbia University

Questions

- Maximum QA transform
- Proximity of QA & QH

Outline

- Near Axis Expansion & QuasiSymmetry
- Ellipticity & Transform
- Simple Model

How can we gain insight into QS?

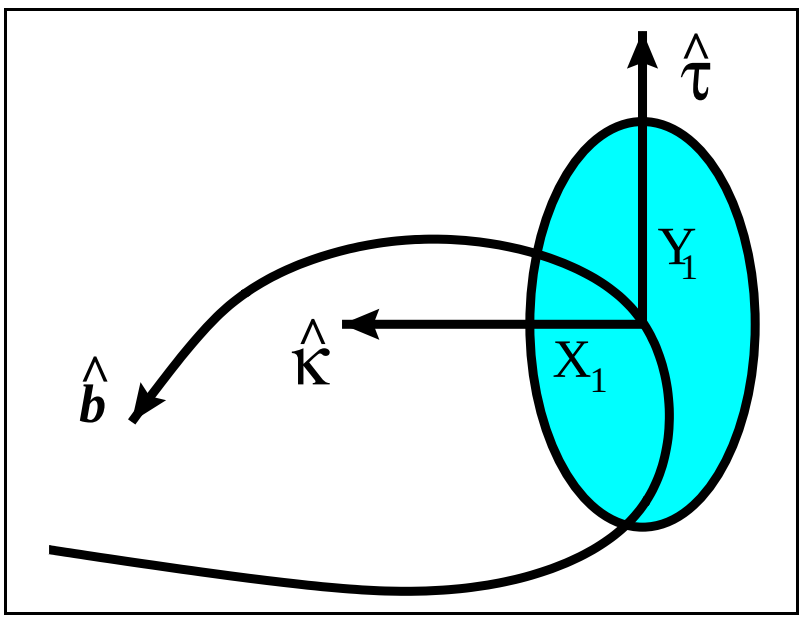
- Near Axis Expansion
  - QuasiSymmetry very restrictive near axis
  - 1st order: ellipticity & transform
  - 2nd order: triangularity & shear
  - closely tied to axis parameters:
    transform requires curvature & torsion

Near Axis Expansion

- Near Axis Expansion $\rho \sim \sqrt{\psi}$

$$\mathbf{X} = \mathbf{X}_0(\ell) + \rho X_1 \hat{\kappa}(\ell) + \rho Y_1 \hat{\tau}(\ell)$$

$$B = B_a(\ell)(1 + \rho\kappa X_1 + \ldots) \qquad (1)$$



- Frenet Equations: define the axis

$$\frac{d\mathbf{X}_0}{d\ell} = \hat{\mathbf{b}}, \qquad \hat{\tau} = \hat{\mathbf{b}} \times \hat{\kappa}$$

$$\frac{d\hat{\mathbf{b}}}{d\ell} = \kappa(\ell)\hat{\kappa}, \quad \frac{d\hat{\kappa}}{d\ell} = -\kappa(\ell)\hat{\mathbf{b}} - \tau(\ell)\hat{\tau}, \quad \frac{d\hat{\tau}}{d\ell} = \tau(\ell)\hat{\kappa}$$

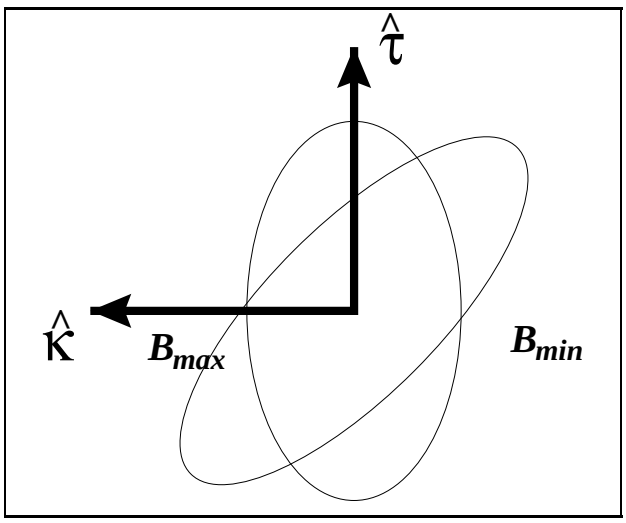
1st Order QuasiSymmetry

$$B = B_a(1 + \delta \cos\theta_H + \ldots) \qquad (2)$$

$$\theta_H = \theta - N\phi$$

θ poloidal, ϕ toroidal magnetic angles

- On axis, $B_a = const \rightarrow \phi = 2\pi\ell/L_a$
  θ_H set to curvature direction



- Surfaces tied to curvature $\kappa X_1 = \delta \cos\theta_H$

- $B_a = const \rightarrow$ Flux=Area Conservation
  $Y_1 = (\frac{\kappa}{\delta})(\sin\theta_H + \sigma\cos\theta_H)$

"Wobble" Function σ

ellipses must "wobble" to keep field lines straight in magnetic coords

- σ rotates ellipses and raises ellipticity

- Stellarator Symmetry $\rightarrow$
  $\sigma = 0$ in principle planes

- Ellipticity in principle planes $E = (\kappa/\delta)^2$

- σ can't "turn over" ellipses
  (maximum rotation 45°)

- Choosing δ can make ellipses rotate relative to curvature direction

Transform

- Transform determined via Ricatti eq
  $\frac{d\sigma}{d\phi} = 2\frac{L_a}{2\pi}\tau(\phi)(\frac{\delta}{\kappa})^2 - \iota_H\{1 + (\frac{\delta}{\kappa})^4 + \sigma(\phi)^2\}$
  where $\iota_H = \iota - N$

- Periodicity condition $\sigma(0) = \sigma(2\pi) \rightarrow$ unique eigenvalue ι_H for each σ_0

- Stellarator Symmetry $\rightarrow \sigma_0 = 0$

- Upper Bound (on magnitude)

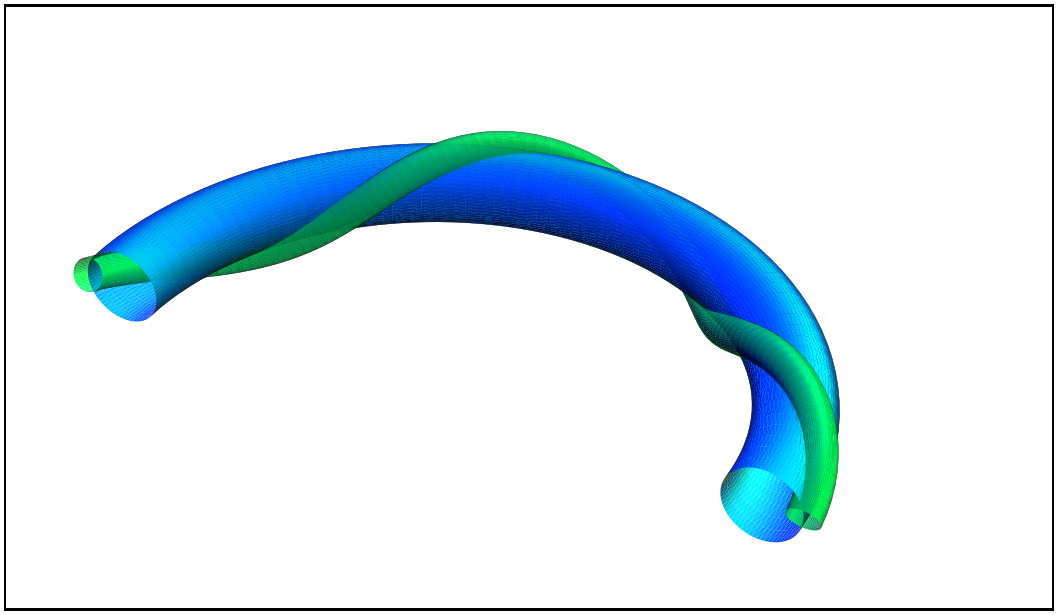
$$\iota_H = \frac{L_a}{2\pi}\frac{\int \tau(\frac{\delta}{\kappa})^2 d\phi}{\int 1+(\frac{\delta}{\kappa})^4 d\phi/2} \sim \frac{2}{E^{-2}+E^2}\frac{1}{2\pi}\int \tau d\ell$$

- Transform has maximum in ellipticity
  $\kappa_{min} < \delta < \kappa_{max}$

Simple Model for Axis

- wind axis on torus of circular x-section
  cylindrical coordinates $\{R, \Phi, Z\}$

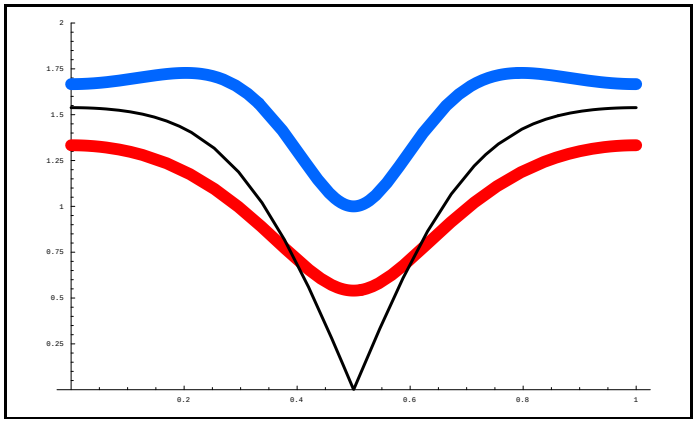
  $R = 1 + a \cos N\Phi$
  $Z = a \sin N\Phi$



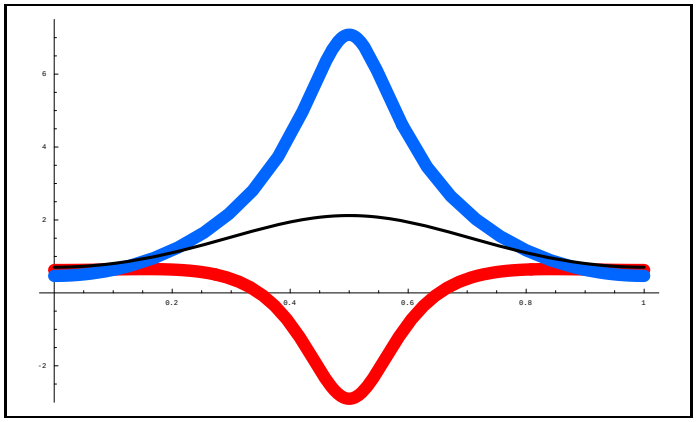
- Critical value $a = 1/(N^2 + 1)$
  $\kappa = 0$ separates QA from QH
  $N = 3 \rightarrow a_{crit} = 0.1$

Curvature & Torsion
$(N = 3, a/a_{crit} = 1/2, 1, 2)$

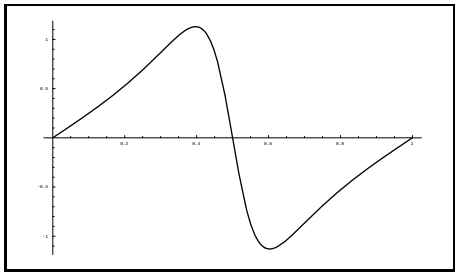
- Curvature



- Torsion



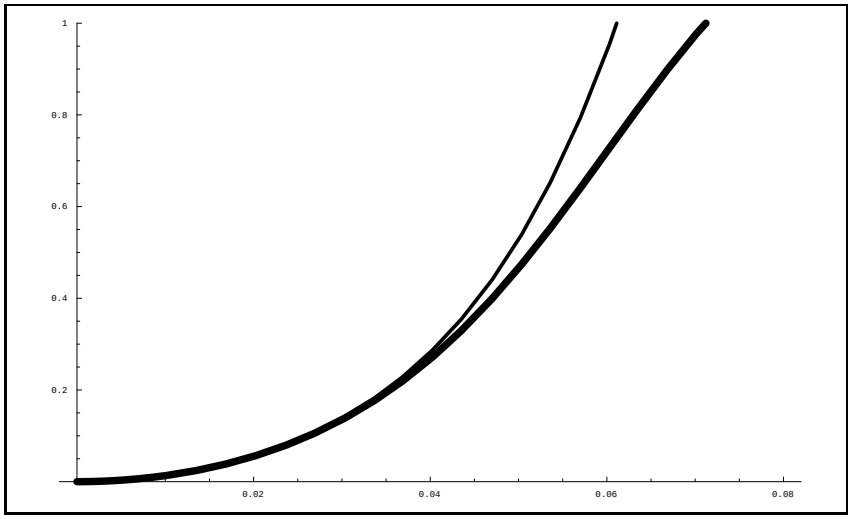
- $\sigma, a/a_{crit} = 1/2, \iota = 0.52$



Transform (N=3 QAS)

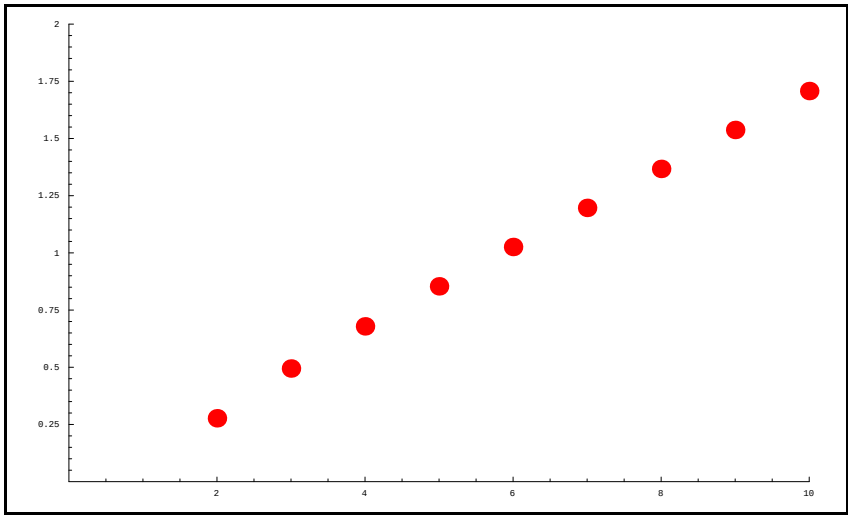
- Transform rises quadratically in a
  upper bound good for small a

ι v. a for $E_0 = 2$



- QAS ι rises linearly with N
  Roughly $\iota = N/6$ @ $a/a_{crit} = 1/2$ & $E_0 = 2$

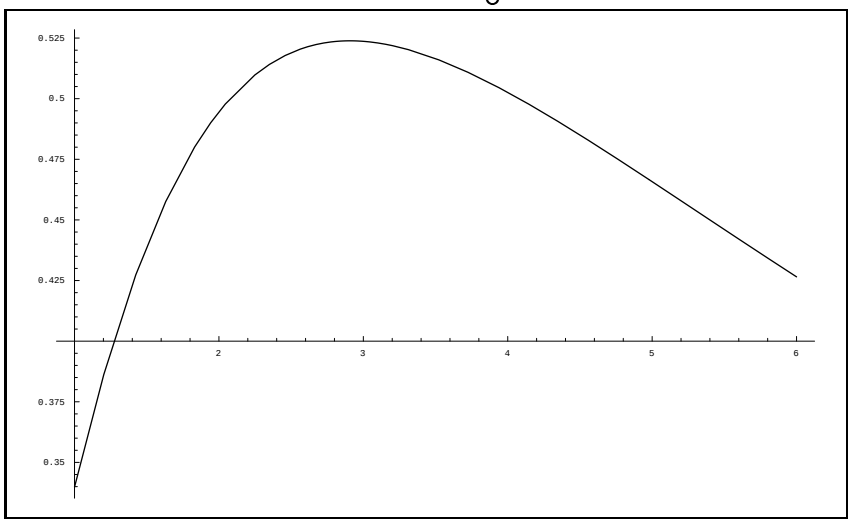
ι v. N



Transform (N=3 QAS)

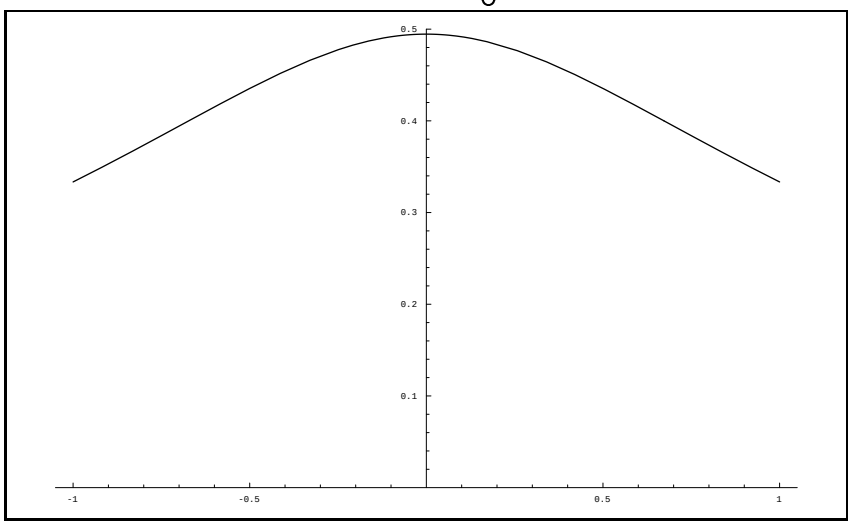
- Has maximum in ellipticity

ι v. E_0



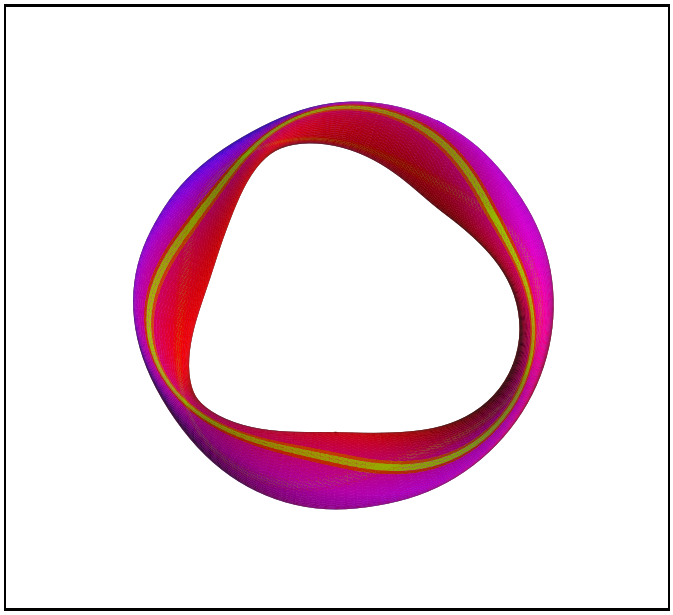
- $\sigma_0 = 0$ gives best ι
  and preserves Stellarator Symmetry

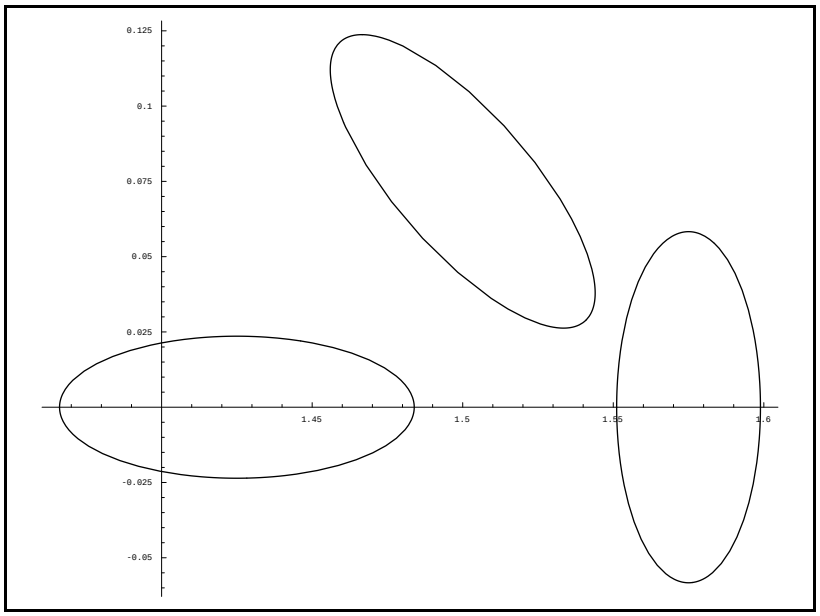
ι v. σ_0



N=3 QAS

$$a/a_{Crit} = 0.5,\ \iota = 0.52\ ,\ E \simeq 2.5$$

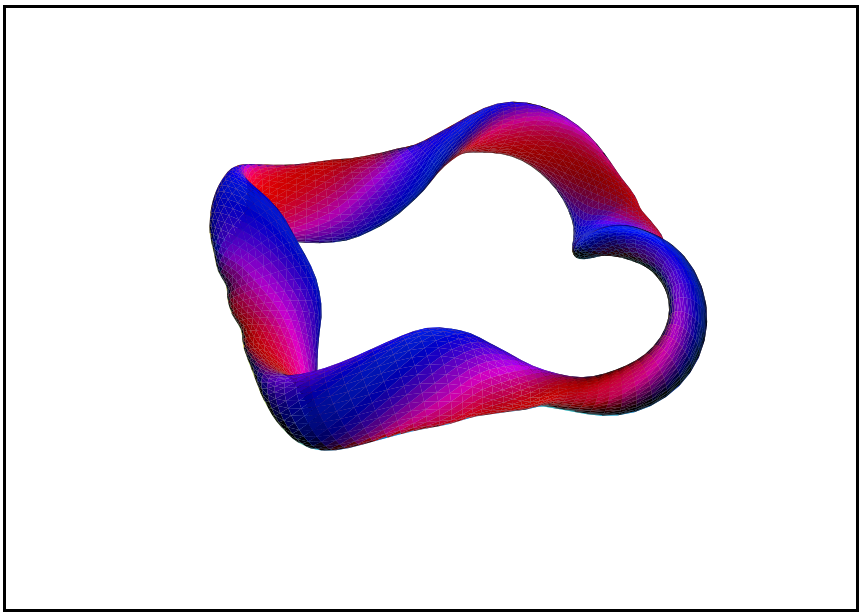


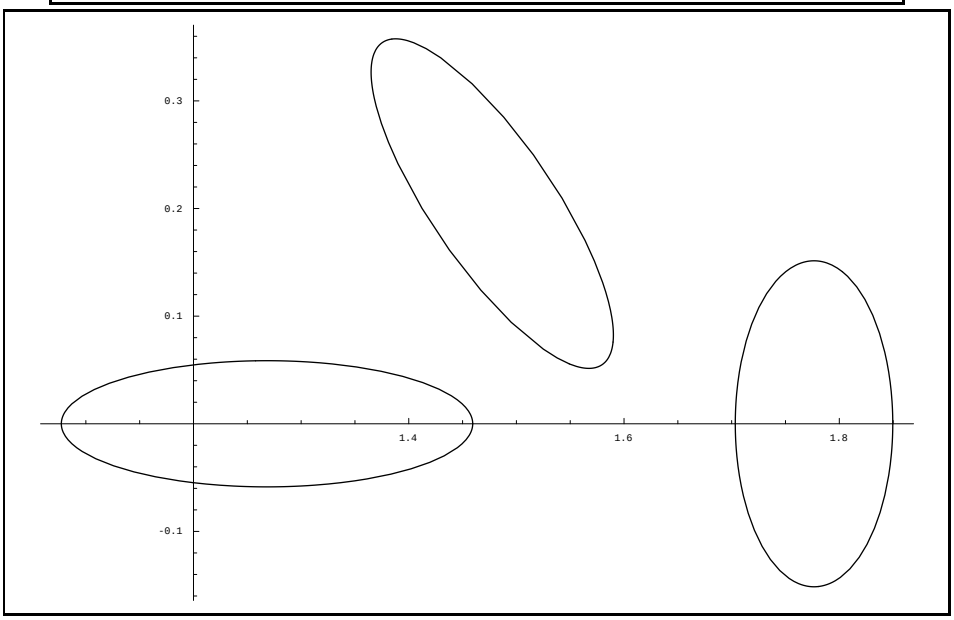


QHS

Joe Talmadge kindly gave HSX axis

$a/a_{crit} \sim 2.5, \iota_{axis} = 1.05, E_{axis} = 2.36$





Summary

- Exact result on axis:
  $\iota = 0.5$ achievable $N = 3$ QAS

- QAS & QHS similar when surfaces rotate
  difference is $a/a_{crit} \sim 1/2, 2$

  $N = 3,\ R_a = 1.5m \rightarrow$
  QAS $a \sim 7.5cm$, QHS $a \sim 30cm$

- Optimize from near axis surfaces

- Choose $\Delta_{1,n}$ terms to mimick axis

Reading List: Quasi-Symmetry Near Axis

- discovery of quasi-helically symmetric systems

  Nührenberg & Zille, Phys. Let. A **129** 2, 113-7

- near axis expansion

  Garren & Boozer, Phys. Fluids B, **3** 10, 2805-21

- 1st order quasihelical symmetry

  Garren & Boozer, Phys. Fluids B, **3** 10, 2822-33

- 2nd order QHS & Mercier stability

  Isaev & Mikhailov, Plasma Phys. Rep, **22** 1, 43-48

- 3D QHS optimization & stability study using approximation of near-axis surfaces

  Isaev, Cooper, et. al., Nuc. Fusion, **37** 10 1431-6