Princeton University: PPPL Mechanical Engineering Division

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Subject: Effects of Axis Excursion on Rotational Transforms

A. Introduction

Ilon Joseph and Allen Boozer (J&B), using near axis expansion and quasi-symmetry constraint, concluded that significant amount of rotational transform may be generated by the helical excursion of the magnetic axis. They showed that (1) for a 3-period configuration, the minor radius of axis excursion, *a*, relative to the major radius, should be less than 0.1; (2) the transform generated by the axis helical excursion increases quadratically with *a*; and (3) for an excursion minor radius half of the critical value, the largest transform occurs when the near axis ellipticity approaches 3. The achievable transform is as high as 0.52. Our earlier attempt to increase the on-axis rotational transform by means of axis wobble alone, however, failed to raise the on-axis transform to > 0.3. Thus, our results are not entirely consistent with the analytic calculations, although we recognize the fact that the configurations used in our study were complex and that there was no effort made to re-optimize the configurations to be quasi-symmetric. In addition, we observed that the resulting configurations became much less stable to MHD modes, both local and global, owing in part to the deterioration of the magnetic well.

To understand the amount of transform that can be generated by helical excursion of the axis and by rotating an ellipse, we carried out a series of numerical calculations using a simple model stellarator in which the ellipicity and axis excursion were varied independently without requiring quasi-symmetry. We only limited the maximum axis excursion to less than 10% of the major radius, which is the critical value as asserted in J&B's work. Although this kind of rotational transform data must exist, it is straightforward to generate them by a series of VMEC calculations.

B. Model Stellarator

We use the following representation for the boundary of our model stellarator,

$$\mathrm{R} + i\mathrm{Z} = e^{iu} \sum \Delta_{m,n} e^{-imu+inv} \quad , \tag{1}$$

where R, Z and $\phi = v/\mathrm{N}$ are cylindrical coordinates about the principal axis of the stellarator, N is the number of field periods, and m, n are the poloidal and toroidal mode numbers, respectively. In the simplest configuration, we retain only four terms: $\Delta_{1,0}$, the major radius; $\Delta_{0,0}$, the minor radius; $\Delta_{2,1}$, the helical twist generating l=2 transform; and $\Delta_{1,1}$, the axis excursion. In a more familiar Fourier notation:

$$R = \Delta_{1,0} + \Delta_{0,0}\cos\theta + \Delta_{2,1}\cos(\theta - N\phi) + \Delta_{1,1}\cos N\phi \qquad (2)$$

$$Z = \Delta_{0,0}\sin\theta - \Delta_{2,1}\sin(\theta - N\phi) - \Delta_{1,1}\sin N\phi \qquad (3)$$

We normalize $\Delta_{0,0}$ to 1, so that $\Delta_{1,0}$ is the aspect ratio, A. Also, we use N=3 throughout the study. This simple stellarator closely resembles the one studied by J&B.

To study the effects of higher order plasma shaping on rotational transform, we also add axisymmetric elongation, $\Delta_{2,0}$, and triangularity terms: $\Delta_{3,0}$, $\Delta_{-1,0}$ (axisymmetric components), and $\Delta_{-1,-1}$ and $\Delta_{3,1}$ (helical components) to the simple model above.

We note that the "effective" helical excursion of the axis determined by equilibrium calculations is somewhat different from $\Delta_{1,1}$, particularly if there are higher order shaping terms. Nevertheless, they are proportional to each other.

C. Rotational Transform (ι)

C.1 ι versus ellipticity and axis excursion.

In Fig. 1, we plot ι on the axis as a function of normalized axis excursion, $\Delta_{1,1}/\Delta_{1,0}$, for three ellipses corresponding to ellipticities 1.7, 3.0 and 7.0, respectively, at A=3. In Fig. 2, we plot ι at the plasma boundary of the same three ellipses. For lower ellipticities, the helical excursion of the axis is more effective to generate transform, but to raise ι above 0.5 wobbling the axis alone is insufficient. On the other hand, for ellipticities large enough to generate high transform axis wobble virtually has no effect on the transform; ι is essentially due to the rotation of the ellipse. The axis wobble merely makes the magnetic well shallower or the magnetic hill steeper. To generate on-axis transform >0.5, the ellipticity needs to be > 3, making the ellipse quite elongated.

We note that for ellipticity ~3 and $\Delta_{1,1}/\Delta_{1,0}$~0.05, the on-axis ι is approximately that calculated by J&B, but the transform of our results is due to the ellipse rotation, not the helical excursion of the axis

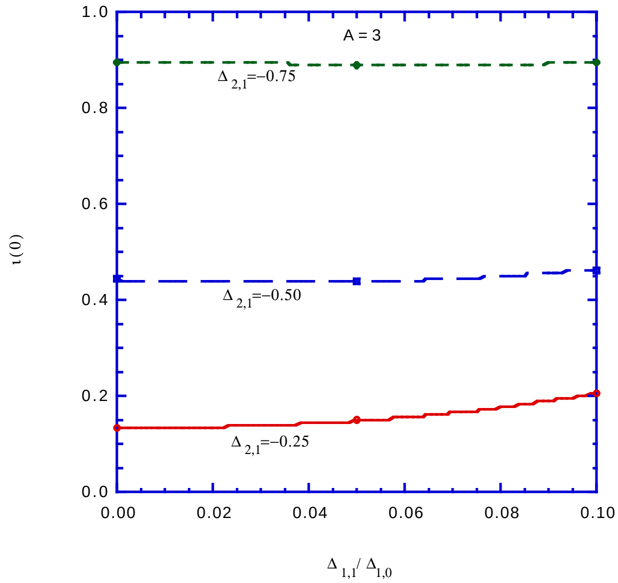

Fig. 1. ι on axis versus axis excursion for ellipticities 1.7, 3.0 and 7.0 and for aspect ratio A=3.

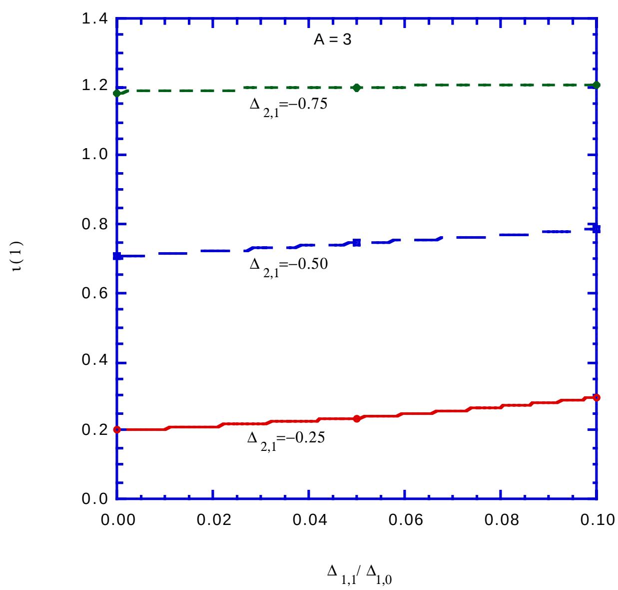

Fig. 2. ι at boundary versus axis excursion for ellipticities 1.7, 3.0 and 7.0 and for A=3.

C.2 Effects of aspect ratio.

Fig. 3 shows the ι generated with various ellipticity and axis excursion for an aspect ratio A= 6 configuration. For the same $\Delta_{1,1}/\Delta_{1,0}$ and ellipticity, the on-axis ι is higher for larger A. Again, it is weakly dependent on the wobble of the axis. The magnetic well property becomes better when compared to a similar configuration with A=3.

C.3 Effects of axisymmetric elongation.

Fig. 4 compares ι for A=3 configurations with three axisymmetric elongations, κ=0.0, 1.5 and 2.0 (corresponding to $\Delta_{2,0}$=0.0, -0.2 and –0.333) added to the rotating ellipses whose ellipticities are 1.7 and 3.0, respectively. The overall ellipticity (E_0) now becomes

$$E_0 = \frac{1.0 - \Delta_{2,0} - \Delta_{2,1}}{1.0 + \Delta_{2,0} + \Delta_{2,1}}\ . \tag{4}$$

The axisymmetric elongation reduces the rotational transform and makes the dependence on axis wobble even weaker.

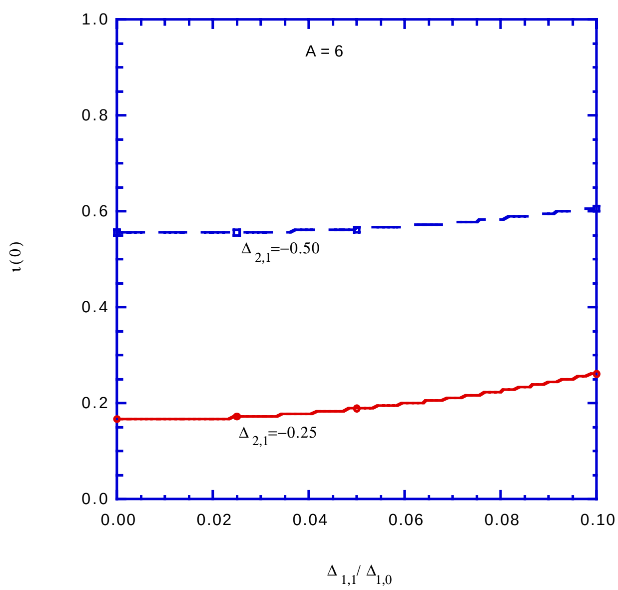

Fig. 3. ι on axis versus axis excursion for ellipticities 1.7 and 3.0 and for aspect ratio A=6.

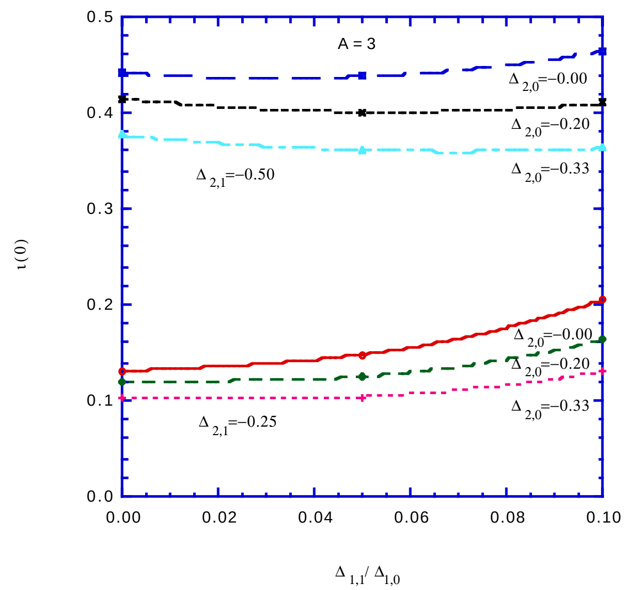

Fig. 4. ι on axis versus axis excursion with axisymmetric elongation added.

C.4 Effects of shaping using triangularity.

Fig. 5 shows the effects on on-axis ι when the second order shaping terms are added. Here, results for A=3 and axisymmetric elongation $\Delta_{2,0}$ =-0.333 (or κ=2) are given. Adding axisymmetric triangularity to an otherwise elliptical configuration generally reduces ι (here we chose $\Delta_{3,0}=\Delta_{-1,0}=0.09$, corresponding to δ=0.5), whereas adding helical trangularity increases ι due to the increase in the effective axis excursion (here, we chose $\Delta_{-1,-1}=0.15$ and $\Delta_{3,1}=-0.03$ as an example). Fig. 6 compares Poincare sections of two configurations with and without triangular shaping.

Fig. 7 shows the variation of edge ι due to the second order shaping. It is seen that the helical triangularity has lesser an effect. For high ellipticity it may even reduce ι. In all the cases the dependence on axis wobble is weak.

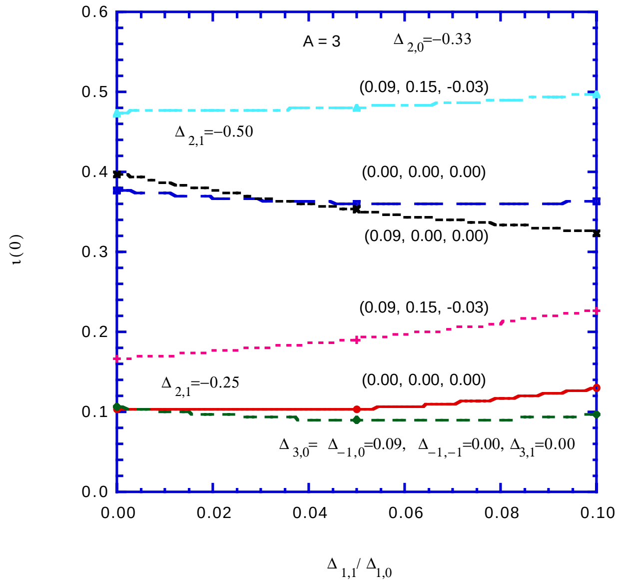

Fig. 5. ι on axis versus axis excursion with triangular shaping added. The three numbers in the parenthesis correspond to $\Delta_{3,0}$, $\Delta_{-1,-1}$ and $\Delta_{3,1}$, respectively.

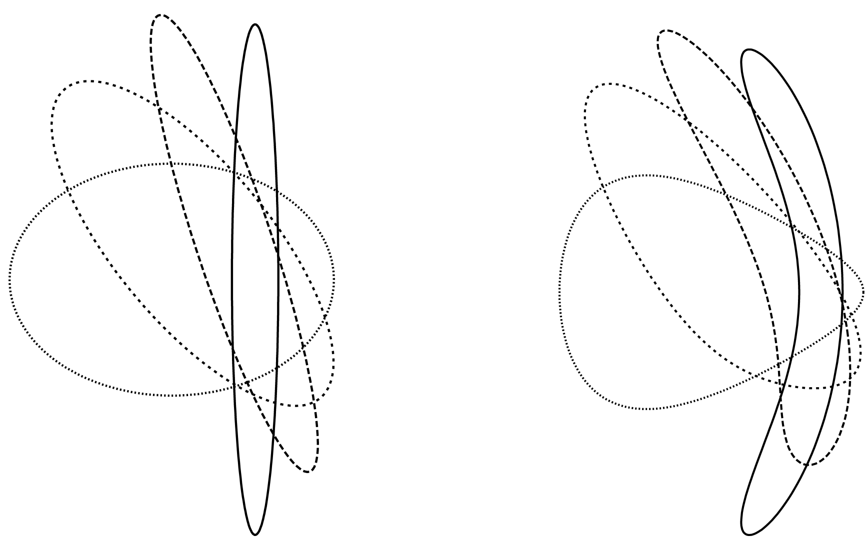

Fig. 6. Four Poincare sections spaced at equal intervals over half a field period for configurations with $\Delta_{2,0}$=-0.33, $\Delta_{2,1}$=-0.50, $\Delta_{1,1}$=0.1·$\Delta_{1,0}$ on the left and with additional $\Delta_{3,0}$=$\Delta_{-1,0}$=0.09 and $\Delta_{-1,-1}$=0.15, $\Delta_{3,1}$=-0.03 on the right.

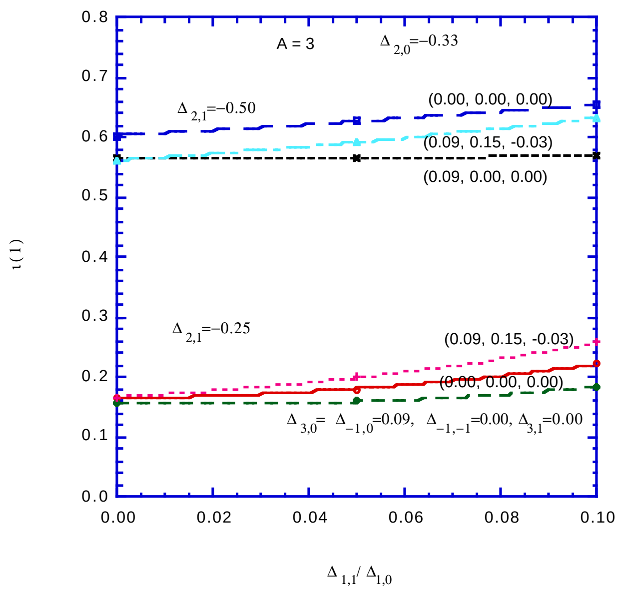

Fig. 7. ι at boundary versus axis excursion with triangular shaping added. The three numbers in the parenthesis correspond to $\Delta_{3,0}$, $\Delta_{-1,-1}$ and $\Delta_{3,1}$, respectively.

C.5 Effects of relative radial direction of axis excursion.

The above discussions assumed $\Delta_{1,1}>0$ so that the axis wobbles outward when an ellipse in vertical position starts to turn. Fig. 8 illustrates the effects of axis wobble when the relative phase is changed such that the axis wobbles inward first ($\Delta_{1,1}<0$). In this figure, only the shaping of the lowest order is included. It shows that the axis excursion has a stronger effect, especially for larger ellipticities. C82 has, in fact, this kind of axis excursion. Still, most of the transform is generated by the rotating ellipse rather than the helical movement of the axis.

The magnetic well property improves with an initial inward axis wobble when compared to an outward wobble. On the other hand, the $B_{1,1}$ component becomes significantly larger and $B_{1,0}$ (m=1, n=0) also becomes more negative, resulting in a deeper main toroidal ripple well.

Fig. 9 shows the effects of the relative phase change of the axis excursion when some of the second order shaping terms are added. We see now that the inward axis wobble has virtually no effect on the rotational transform.

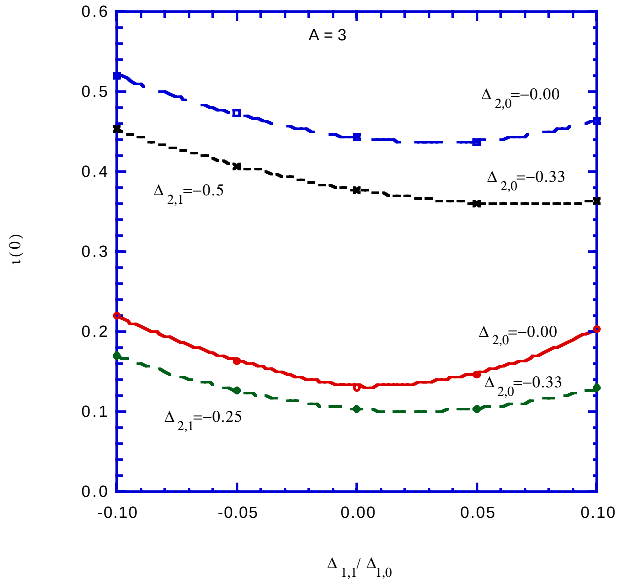

Fig. 8. Comparison of ι on axis for positive and negative $\Delta_{1,1}$.

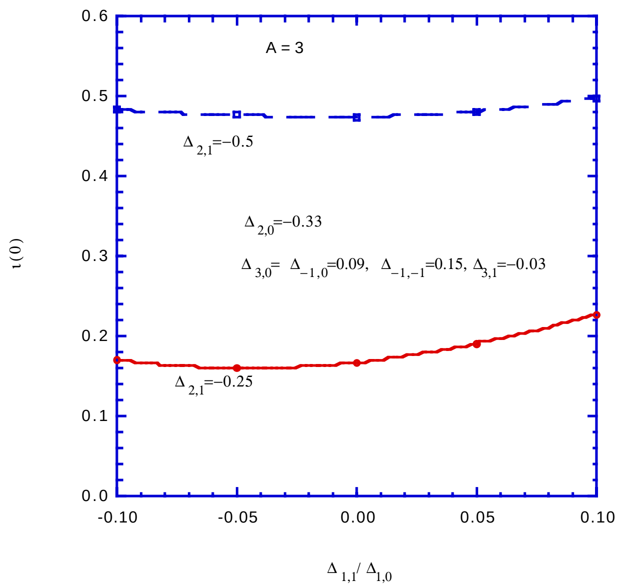

Fig. 9. Comparison of ι on axis for positive and negative $\Delta_{1,1}$ and with second order shaping.

C.6 Effects of relative vertical direction of axis excursion.

The relation we established in Eq. 1 is such that the axis wobbles outward and downward as an ellipse in vertical position starts to turn if $\Delta_{1,1}$ is positive. Fig. 10 compares the on-axis ι when the vertical wobble of the axis is directed initially upward instead. The reversal of the initial vertical direction only makes the dependence of ι on the axis excursion even weaker.

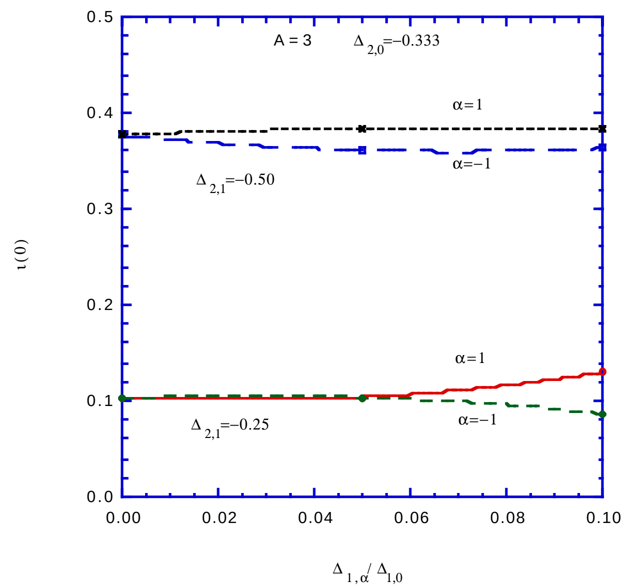

Fig. 10. Comparison of ι on axis for initial vertical direction moving downward (α=1) and upward (α=-1).

D. Conclusions.

VMEC equilibrium calculations using simple model stellarators show that most of the on-axis rotational transform is generated by the rotation of the plasma rather than by the helical excursion of the magnetic axis as long as the excursion radius is limited to < 10% of the major radius. It requires an ellipticity on the order of 3 to generate an on-axis ι ~ 0.5. If, for MHD stability considerations, we also require an axisymmetric elongation on the order of 2, the plasma half width at the narrow section for a 1-m major radius configuration would be

$$a_{\min} = \frac{1+\Delta_{2,0}+\Delta_{2,1}}{\Delta_{1,0}} = \frac{1-0.333-0.5}{\mathrm{A}} = \frac{0.17}{\mathrm{A}} \quad . \qquad (5)$$

Thus, for an aspect ratio 3 machine, the smallest half width may be as small as 0.06 m. Furthermore, to ensure a vacuum magnetic well with an ellipticity ~3, the aspect ratio probably would have to be greater than 3, making the smallest half width even smaller. This is a fact we have to live with if high ι configurations are indeed more desirable from the viewpoint of better particle confinement and smaller bootstrap current.