Rotation Damping in Stellarators

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Outline

1. Introduction
2. Fluid Equations
3. δf Code and Kinetic Equation
4. Damping Coefficient
5. Ambipolar Field

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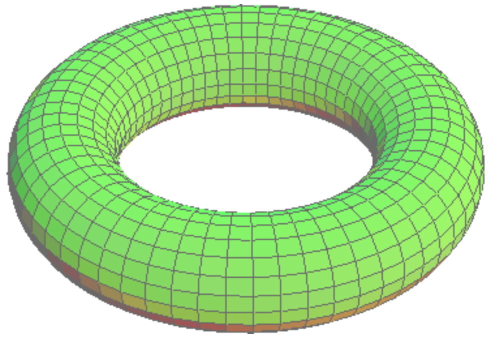
Introduction

- *Importance of Rotation*
  - Transport barriers
  - Improved plasma performance
- *Prediction of Rotation Rate*
  - In toroidal symmetry:

$$\tau_\phi \to \tau_i \left(\frac{a}{\rho}\right)^2 \tag{1}$$

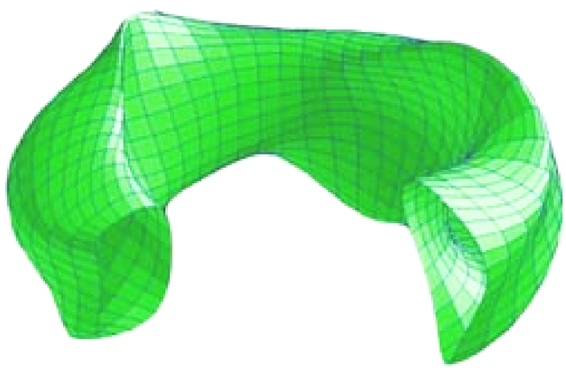
  - In arbitrary stellarator, $\tau_\phi \to \tau_i$
  - Use δf code to estimate the flow
  - Momentum conserving operator causes difficulties

Symmetric



$$\frac{|\mathbf{B}|}{B_0} = 1 + \epsilon_{10} \cos(\theta)$$

Asymmetric



$$\frac{|\mathbf{B}|}{B_0} = 1 + \sum \epsilon_{mn} \cos(m\theta - n\phi)$$

Canonical toroidal momentum, L_ϕ no longer conserved

$$\vec{L} = \vec{p} + e\vec{A}$$

Fluid Equations

Force-Balance Equation:

$$\boxed{\rho \frac{d\vec{\mathbf{u}}}{dt} = -\vec{\nabla} \cdot \overleftrightarrow{\mathbf{P}} + en\left(\vec{E} + \vec{\mathbf{u}} \times \mathbf{B}\right) + \vec{\mathbf{F_d}} - \vec{\mathbf{R}} \qquad (2)}$$

$\vec{\mathbf{F_d}} \rightarrow$ applied force

$\vec{\mathbf{u}} \rightarrow$ flow velocity of the fluid

$\overleftrightarrow{\mathbf{P}} \rightarrow$ pressure tensor, given by:

$$\overleftrightarrow{\mathbf{P}} = p_{||}\hat{\mathbf{b}}\hat{\mathbf{b}} + p_{\perp}(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) \qquad (3)$$

- Want τ such that

$$m\vec{u} = \overleftrightarrow{\tau} \cdot \vec{F_d} \qquad (4)$$

$$\mathbf{B} = \frac{1}{\mathcal{J}}\left(\frac{d\vec{\mathbf{x}}}{d\phi} + \iota\frac{d\vec{\mathbf{x}}}{d\theta}\right) = G\vec{\nabla}\phi + I\vec{\nabla}\theta + \beta_*\vec{\nabla}\psi \quad (5)$$

- $\mathcal{J}$ is the Jacobian , $\mathcal{J} = \mu_o\frac{G+\iota I}{B^2}$
- Net applied toroidal torque is:

$$\mathcal{T}_\phi \equiv \oint \frac{\partial\vec{x}}{\partial\phi}\cdot\vec{F}\mathcal{J}d\theta d\phi \quad (6)$$

- $\vec{\mathbf{u}} \times \mathbf{B}$ integral proportional to the net flow of charge across a surface

$$en\oint\vec{u}\times\mathbf{B}\cdot\frac{d\vec{\mathbf{x}}}{d\phi}\mathcal{J}d\theta d\phi = en\oint\iota\vec{u}\cdot\vec{\nabla}\psi\mathcal{J}d\theta d\phi = \iota\frac{dQ}{dt}$$

Likewise, $en\oint\vec{u}\times\mathbf{B}\cdot\frac{d\vec{\mathbf{x}}}{d\theta}\mathcal{J}d\theta d\phi = -\frac{dQ}{dt}$

- For the pressure tensor, after much algebra:

$$\oint \frac{d\vec{\mathbf{x}}}{d\phi} \cdot \vec{\nabla} \cdot \overleftrightarrow{\mathbf{P}} \mathcal{J} d\theta d\phi = \oint \frac{1}{2} \frac{\partial \left(p_\perp + p_\|\right)}{\partial \phi} \mathcal{J} d\theta d\phi \tag{7}$$

- Canonical Angular Momentum:

$$\vec{L} = \vec{p} + e\vec{A}, \vec{A} = \psi \vec{\nabla}\theta - \psi_p \vec{\nabla}\phi$$

then,

$$\vec{L} \cdot \frac{d\vec{\mathbf{x}}}{d\phi} = \rho \vec{u} \cdot \frac{\partial \vec{\mathbf{x}}}{\partial \phi} - e\psi_p,$$

- Surface integral of ϕ component:

$$\frac{d\mathcal{L}_\phi}{dt} \equiv \oint \rho \frac{d\vec{u}}{dt} \cdot \frac{\partial \vec{\mathbf{x}}}{\partial \phi} \mathcal{J} d\theta d\phi - \iota \frac{dQ_c}{dt} \tag{8}$$

- ϕ component of the fluid equation is then:

$$\boxed{\frac{d\mathcal{L}_\phi}{dt} = \mathcal{T}_\phi - \oint \frac{1}{2} \frac{\partial \left(p_\perp + p_\|\right)}{\partial \phi} \mathcal{J} d\theta d\phi} \tag{9}$$

The δf method

- The Fokker-Planck equation,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{q}{m}\left(\vec{E} + \vec{v} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial \vec{v}} = C\left(f\right) \tag{10}$$

- Assume the distribution function, f, is approximately a Maxwellian.

$$f = F_m\left(1 + \delta\right) \tag{11}$$

- The Maxwellian has the form

$$F_m\left(\psi, H\right) = \exp\left[\zeta\left(\psi\right) - 1 - H\beta\left(\psi\right)\right] \tag{12}$$

- The Hamiltonian, H, is

$$H = \frac{\left(2\pi\right)^2}{2m}\left(\frac{B}{\mu_o G}\right)^2\left(p_\phi + \frac{q\psi_p}{2\pi}\right)^2 + \mu B \tag{13}$$

- Fokker Planck Equation:

$$\frac{d\delta}{dt} + \vec{v_g} \cdot \vec{\nabla}\psi \frac{\partial \ln F_m}{\partial \psi} + \vec{F_d} \cdot \frac{\partial F_m}{\partial \vec{v}} = C_m(\delta) \tag{14}$$

- Applied forces

$$\vec{F_d} = -q\frac{d\Phi}{d\psi}\vec{\nabla}\psi + f_\phi \vec{\nabla}\phi + f_\theta \vec{\nabla}\theta \tag{15}$$

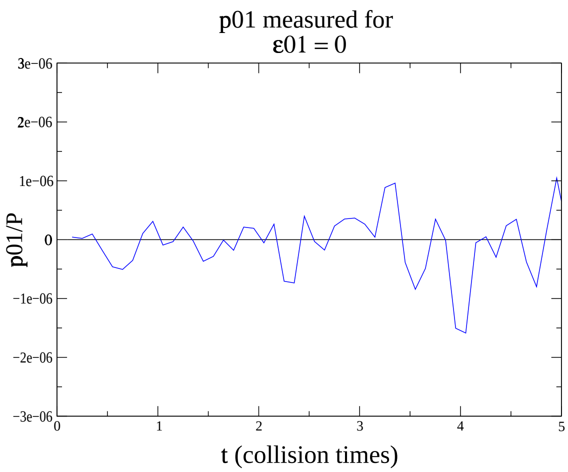
- During each time-step
  - Hamiltonian step

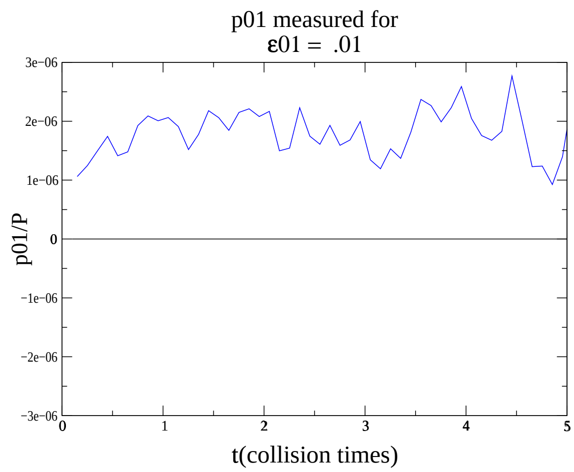
    &
  - Collision step
- Particles are restricted to narrow ψ-annulus

- Calculation using $\delta_i' s$

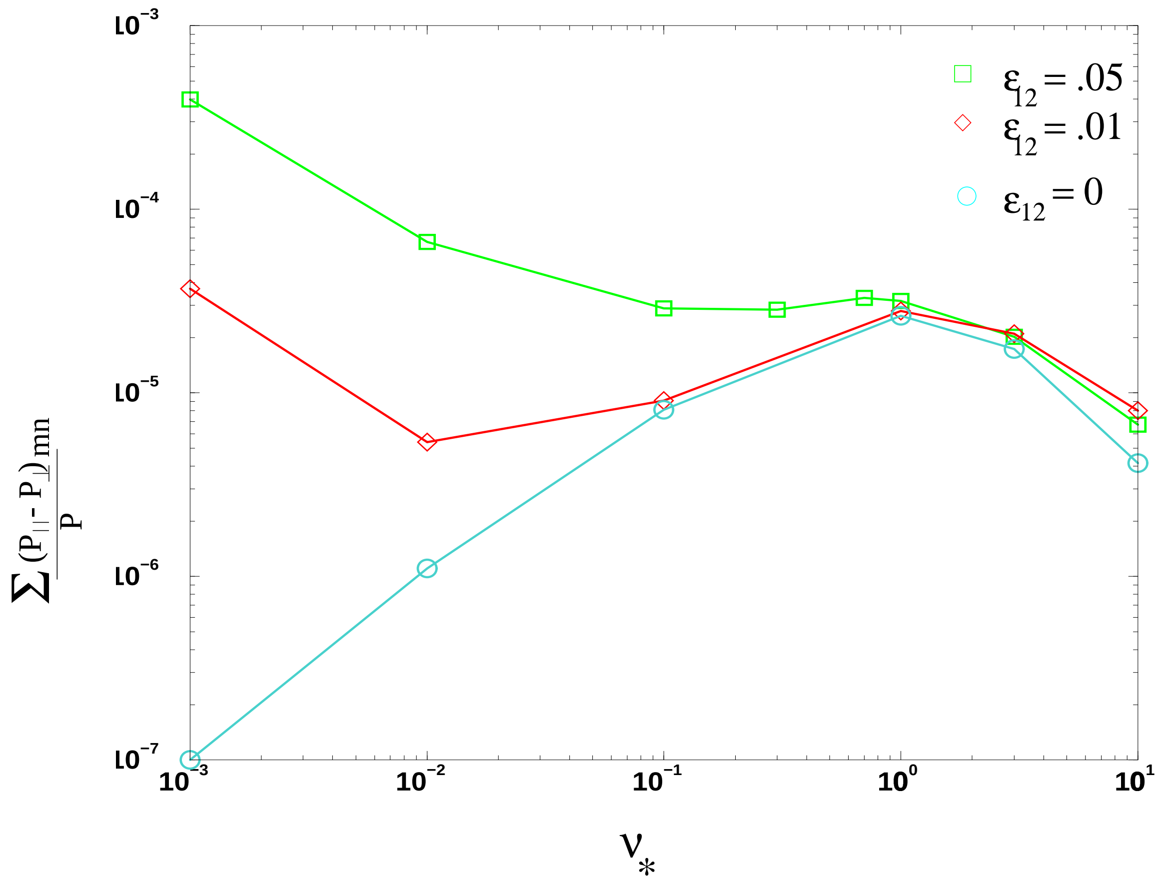
$$p_\perp = \sum_{m,n} (p_\perp)_{m,n}\, e^{2\pi i(n\phi - m\theta)} \tag{16}$$

$$\boxed{\boxed{\begin{aligned}(p_\perp)_{mn} &= \int \frac{1}{2} m {v_\perp}^2 f e^{-2\pi i(n\phi - m\theta)} d^3 v\, d\theta\, d\phi \\ &= -\frac{m{v_0}^2 \frac{dn}{d\psi} \sum_i \left[\left(1 - {\lambda_i}^2\right) \delta_i {B_i}^2 e^{-2\pi i(n\phi_i - m\theta_i)}\right]}{2\sum_i {B_i}^2}\end{aligned}}}$$





Viscosity calculation with δf code:



Analytic prediction at high collisionality:

$$\left(P_{\|} - P_{\perp}\right) \approx \frac{3}{2}\frac{nT}{\nu}\left[\hat{\mathbf{b}} \cdot \vec{\nabla}\left(\hat{\mathbf{b}} \cdot \vec{v}\right) - \left(\hat{\mathbf{b}} \cdot \vec{\nabla}\hat{\mathbf{b}}\right) \cdot \vec{v}\right] \tag{17}$$

(Grimm and Johnson, Plasma Phys. 14, 615(1972))

- The viscosity is linearly related to $\delta_i's$.
- $\delta_i \propto F_d$ from Fokker-Planck
- Therefore, $p_\perp \propto F_d$, $p_{||} \propto F_d$

$$\oint \frac{1}{2}\frac{\partial\left(p_\perp + p_{||}\right)}{\partial\phi}\mathcal{J}\,d\theta d\phi = \beta_\phi^{(\phi)} f_\phi + \beta_\theta^{(\phi)} f_\theta + \beta_\psi^{(\phi)}\frac{d\Phi}{d\psi}$$

$$\oint \frac{1}{2}\frac{\partial\left(p_\perp + p_{||}\right)}{\partial\theta}\mathcal{J}\,d\theta d\phi = \beta_\phi^{(\theta)} f_\phi + \beta_\theta^{(\theta)} f_\theta + \beta_\psi^{(\theta)}\frac{d\Phi}{d\psi}$$

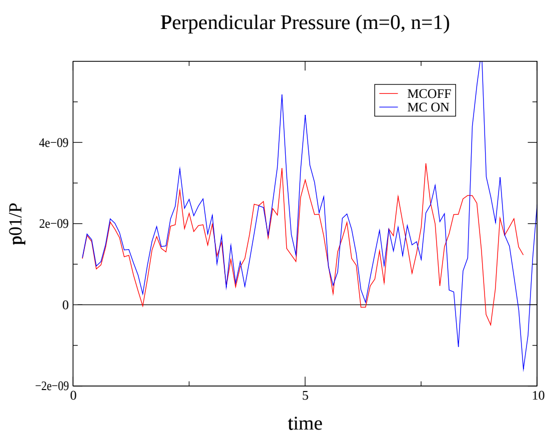
- All $\beta's$ can be calculated with a single simulation
- Hamiltonian only calculated once

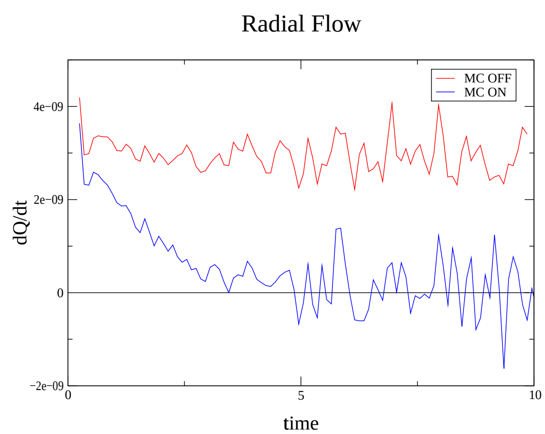
Momentum Conservation

- Modified Lorentz collison operator that conserves momentum:

$$C(f) = \frac{\nu}{2}\frac{\partial}{\partial\lambda}\left[\left(1-\lambda^2\right)\frac{\partial f}{\partial\lambda}\right]$$
$$\longrightarrow \frac{\nu}{2}\frac{\partial}{\partial\lambda}\left[\left(1-\lambda^2\right)\left(\frac{\partial f}{\partial\lambda} - 3\frac{u_{\parallel}}{v}f\right)\right] \tag{18}$$

- $\lambda \equiv \frac{v_{\parallel}}{v}$
- $u_{\parallel}$ in Eq. 18 is averaged over a collision time
- Imperfect momentum conservation has greater effect on $\frac{dQ}{dt}$ than $p_{\perp}$





- In the δf code:

$$\frac{d\delta}{dt} = -\vec{v_g} \cdot \vec{\nabla}\psi \frac{\partial \ln F_m}{\partial \psi} - \frac{m}{T} \vec{F_d} \cdot \vec{v_g} + 2\nu \left(1 - 3\lambda^2\right) \langle \lambda \rangle \tag{19}$$

- In principle, $\langle \lambda \rangle \equiv \int \lambda \delta d\lambda$.
- Let

$$\langle \lambda \rangle \equiv \int \lambda \delta d\lambda + \frac{\Lambda_{\|}}{\oint \mathcal{J} d\theta d\phi} \tag{20}$$

- Acts as additional driving term:

$$\oint \frac{1}{2} \frac{\partial \left(p_{\perp} + p_{\|}\right)}{\partial \phi} \mathcal{J} d\theta d\phi = \beta_{\phi}^{(\phi)} f_{\phi} + \beta_{\theta}^{(\phi)} f_{\theta} + \beta_{\psi}^{(\phi)} \frac{d\Phi}{d\psi} + \beta_{\Lambda}^{(\phi)} \Lambda_{\|} \tag{21}$$

- Now have two Eqns., (θ, ϕ components of the fluid equation), and two unknowns $\left(\Lambda_{\|}, \frac{d\Phi}{d\psi}\right)$

Damping Coefficient

- Have

$$\frac{d\Phi}{d\psi} = \delta\vec{E}\cdot\frac{d\vec{\mathbf{x}}}{d\psi} \propto F_d$$

- Therefore

$$\delta\vec{u}_\perp = \frac{\delta\vec{E}\times\mathbf{B}}{B^2} \tag{22}$$

- Need perturbed parallel flow, $u_{\|} = u_{\|o} + \delta u_{\|}$

$$\delta u_{\|} = \gamma_\phi f_\phi + \gamma_\theta f_\theta + \gamma_\psi\frac{d\Phi}{d\psi} \tag{23}$$

- Therefore:

$$\boxed{m\vec{u} = \overleftrightarrow{\tau}\cdot\vec{\mathbf{F}}_d \qquad (24)}$$

Ambipolarity

- Can also use δf code to find radial ambipolar field.

$$j_r = en\left[\left(u_i\right)_r - \left(u_e\right)_r\right] = \sigma\left(E_r - E_a\right) \tag{25}$$

- Background ambipolar field in the orbit equations
- Adding driving forces, requires additional field
- Separate the perturbed field

$$j_r = \sigma\left(\left(E_r - E_a\right)_0 + \left(\delta E_r - \delta E_a\right)\right) \tag{26}$$

- The perpendicular conductivity is

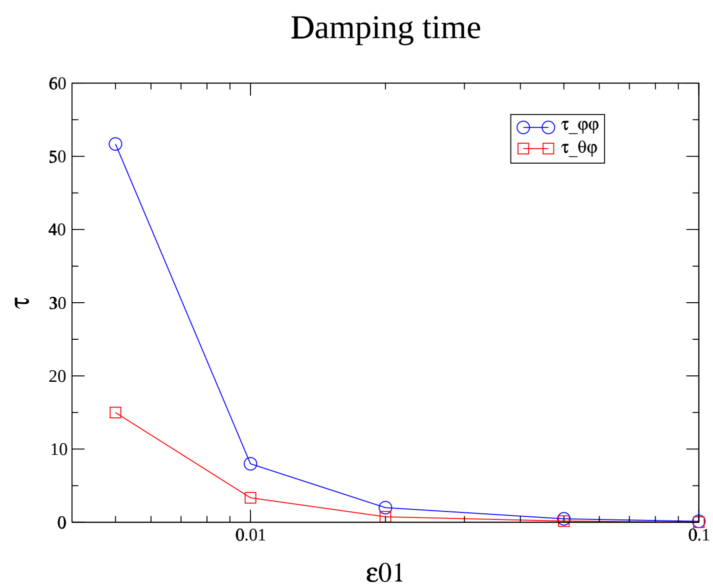
$$\sigma = \frac{\beta_\psi^{(\phi)}\beta_\Lambda^{(\theta)} - \beta_\psi^{(\theta)}\beta_\Lambda^{(\phi)}}{\beta_\Lambda^{(\phi)} + \iota\beta_\Lambda^{(\theta)}} \tag{27}$$

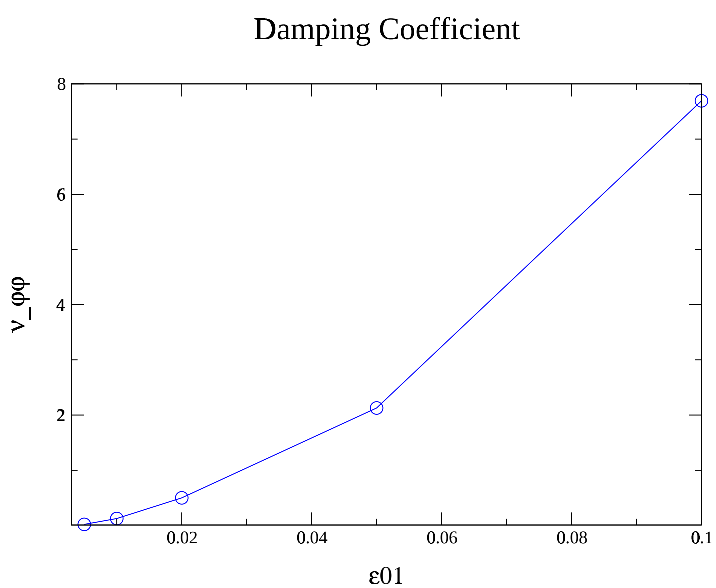
- If momentum conservation were perfect:

$$\sigma = \frac{1}{\iota}\beta_\psi^{(\phi)} = -\beta_\psi^{(\theta)}$$

Calculations

- Found $\tau \propto \left(\frac{1}{\epsilon_{01}}\right)^2$





Summary

- Fluid eqns. improve δf calculations of rotation damping
- Accounted for problems with collision operator in δf
- Used δf and fluid eqns. to find damping coefficient and perturbed $\frac{dQ}{dt}$
- Demonstrated $\sim \frac{1}{\delta^2}$ dependence of τ