OPTIMAL CURRENT POTENTIAL FOR STELLARATOR COILS

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Modern stellarator coils designed by first finding a surface current with desired properties

$$\vec{j} = \delta\left(\vec{X}-\vec{X}_c(\theta,\varphi)\right)\vec{\nabla}\kappa(\theta,\varphi)\times\hat{n}.$$

$\vec{X}_c(\theta,\varphi)$ gives surface on which coils will be located.
$\hat{n}$ the normal to the coil surface.
$\kappa(\theta,\varphi)$ the current potential.

Contours of current potential, κ, give location of wires.
Jump in κ between contours gives current in wires.

Presently κ optimized to support a single plasma.

Properties we wish to impose on $\kappa(\theta,\varphi)$

1. flexibility to support many desirable plasmas.
2. maximum efficiency (minimum Ohmic losses and $\langle j^2\rangle$).
3. space reserved for ports.

BASIC CONCEPTS

Proposed optimization of κ uses:

Flux $\vec{\Phi}$ $$\Phi_i = \int_{\substack{\text{plasma}\\ \text{surface}}} f_i(\theta,\varphi)\vec{B}_o \cdot d\vec{a}$$

$f_i(\theta,\varphi)$ any complete set of functions (like trig functions).

$\vec{B}_o$ field due to all other sources than coils being designed.

Current $\vec{I}$ $$\kappa(\theta,\varphi) = \sum_j I_j\, g_j(\theta,\varphi)$$

$g_j(\theta,\varphi)$ any set of dimensionless functions.

Inductance $\overleftrightarrow{L}$ $$L_{ij}I_j = \int_{\substack{\text{plasma}\\ \text{surface}}} f_i(\theta,\varphi)\, \vec{B}_j \cdot d\vec{a}$$

$\vec{B}_j$ is field produced by current potential $I_j\, g_j(\theta,\varphi)$.

Quality $\overleftrightarrow{Q}$ $$T = T_o - (\vec{\Phi} - \overleftrightarrow{L} \cdot \vec{I})^T \cdot \overleftrightarrow{Q} \cdot (\vec{\Phi} - \overleftrightarrow{L} \cdot \vec{I})$$

T is target function of physics optimization.

Resistance $\overleftrightarrow{R}$ $$P_{ohmic} = \vec{I}^T \cdot \overleftrightarrow{R} \cdot \vec{I}$$

QUALITY MATRIX $\overleftrightarrow{Q}$

Quality matrix determines flux components:

1. that are important to reproduce a single plasma configuration accurately.
2. that must be controlled to if the coil set is to produce other configurations.

Calculation of $\overleftrightarrow{Q}$

Stellarators designed by optimizing a target function $T(a_{mn})$ which gives dependence of physics properties (MHD, drift orbits, etc) on the Fourier components, a_{mn}, of the shape of the plasma surface. (Nührenberg)

Target function near optimum depends roughly quadratically on changes in plasma shape, δa_{mn}

Displacement $\vec{\xi}$ of plasma shape gives a perturbed flux,

$$\int f_i \vec{b} \cdot d\vec{a} = \int (\vec{B} \cdot \vec{\nabla} f_i) \vec{\xi} \cdot d\vec{a} \propto \delta a_{mn}.$$

Therefore: $T = T_o - (\vec{\Phi} - \overleftrightarrow{L} \cdot \vec{I})^T \cdot \overleftrightarrow{Q} \cdot (\vec{\Phi} - \overleftrightarrow{L} \cdot \vec{I})$

Important part of quality matrix $\overleftrightarrow{Q}$

Diagonalize $\overleftrightarrow{Q}$. Eigenmode "i" is important if $\delta Q = q_i \Phi_i^2$ is large with q_i the eigenvalue.

Eigenmode can be important because:

1. eigenvalue q_i is large.
2. associated flux Φ_i is large.

Order the eigenmodes of $\overleftrightarrow{Q}$ by importance.

An acceptable quality stellarator configuration is obtained if a finite number of eigenmodes is retained.

Let N_Φ be the required number of eigenmodes to obtain an acceptable configuration.

Assume $\vec{\Phi}$ contains N_Φ components, namely the required flux components.

RESISTANCE MATRIX $\overleftrightarrow{R}$

Defined by $$P_{\text{ohmic}} = \vec{I}^T \cdot \overleftrightarrow{R} \cdot \vec{I} = \int_{\text{coil surf.}} \eta\, j^2 d^3x$$

Current density $$\vec{j} = \delta\left(\vec{X} - \vec{X}_c(\theta,\varphi)\right) \vec{\nabla}\, \kappa(\theta,\varphi) \times \hat{n}.$$

Current potential $$\kappa(\theta,\varphi) = \sum_{j=1}^{N_I} I_j\, g_j(\theta,\varphi).$$

Finite resistance requires current channel have a finite thickness. Let:

$v(\theta,\varphi)\,\delta\theta\delta\varphi$ current channel volume in the interval $\delta\theta\delta\varphi$.

η resistivity of coil material.

Resistance matrix is

$$R_{ij} = \int \frac{\eta}{v} \left\{ \left(\frac{\partial \vec{X}_c}{\partial \varphi}\right)^2 \frac{\partial g_i}{\partial \theta}\frac{\partial g_j}{\partial \theta} - \left(\frac{\partial \vec{X}_c}{\partial \theta} \cdot \frac{\partial \vec{X}_c}{\partial \varphi}\right)\left(\frac{\partial g_i}{\partial \theta}\frac{\partial g_j}{\partial \varphi} + \frac{\partial g_i}{\partial \varphi}\frac{\partial g_j}{\partial \theta}\right) + \left(\frac{\partial \vec{X}_c}{\partial \theta}\right)^2 \frac{\partial g_i}{\partial \varphi}\frac{\partial g_j}{\partial \varphi} \right\} d\theta d\varphi$$

PROPOSED $\kappa(\theta,\varphi)$ CALCULATION

N_Φ number of important components of the flux. These are to be produced exactly.

N_I number of components of current. ($N_I >> N_\Phi$)

Find minimum Ohmic power required to produce the N_Φ flux components exactly.

Diagonalize $\overleftrightarrow{L}^T \cdot \overleftrightarrow{L}$. This matrix has:

Precisely N_Φ non-zero eigenvalues ℓ_j^2 if coil set can reproduce required fluxes.

N_Φ eigenvectors $|s\rangle$ with non-zero ℓ_j^2.

N_I - N_Φ eigenvectors $|n\rangle$ with zero ℓ_j^2.

Current $\vec{I} = \sum_{s=1}^{N_\Phi} I_s|s\rangle + \sum_{n=1}^{N_I - N_\Phi} I_n|n\rangle$ reproduces the N_Φ fluxes exactly with arbitrary I_n.

Use arbitrary I_n to minimize the Ohmic power

Given by $\sum_{n=1}^{N_I-N_\Phi} R_{n'n} I_n = -\sum_{s=1}^{N_\Phi} R_{n's} I_s$, so

$$I_n = -\sum_{s=1}^{N_\Phi} I_s c_{sn}.$$

The current which exactly reproduces the fluxes with minimal Ohmic power is

$$\vec{I} = \sum_{s=1}^{N_\Phi} I_s \left(|s\rangle - \sum_{n=1}^{N_I-N_\Phi} c_{sn} |n\rangle \right).$$

Gives minimum Ohmic power required to produce any given set of N_Φ fluxes.

Gives set of currents needed for the flexibility to drive the important fluxes independently.

Note: One can reserve area for ports. Make either $g_j(\theta,\varphi)$ = constant or η/v large where ports are to be.

METHOD USED NOW

Minimize (Merkel) $\mathcal{E}^2 = (\vec{\Phi} - \overleftrightarrow{L} \cdot \vec{I})^T \cdot (\vec{\Phi} - \overleftrightarrow{L} \cdot \vec{I})$

Solution is $\overleftrightarrow{L} \cdot \vec{I} = \vec{\Phi}$ but for a generic, simple, smooth plasma boundary $I \equiv \sqrt{\vec{I}^T \cdot \vec{I}} \rightarrow \infty$ even for a small plasma/coil separation.

SVD techniques (Pomphrey) allow one to find the smallest $\mathcal{E}$ consistent with a given current I.

$\overleftrightarrow{L} = \overleftrightarrow{Y} \cdot \overleftrightarrow{\ell} \cdot \overleftrightarrow{Z}^T$ with $\overleftrightarrow{Y}$ and $\overleftrightarrow{Z}$ orthogonal and $\overleftrightarrow{\ell}$ diagonal. Denote eigenvalues by ℓ_j.

Use eigen flux and current $\vec{\Phi}^{(e)} \equiv \overleftrightarrow{Y}^T \cdot \vec{\Phi}$ and $\vec{I}^{(e)} \equiv \overleftrightarrow{Z}^T \cdot \vec{I}$,

$$\Phi_j^{(e)} = \ell_j I_j^{(e)}$$

Solve only those equations "j" with $\ell_j > \ell_{min}$

Error $\mathcal{E}(\ell_{min})$ smaller the smaller ℓ_{min}; current $I(\ell_{min})$ becomes bigger.

Note $N_\Phi >> N_I$ unlike proposed method.

Problems with present $\kappa(\theta,\varphi)$ solver:

1. Fluxes ignored that may be essential.
   poor reconstruction

2. Fluxes retained which are not important.
   $\kappa(\theta,\varphi)$ larger than it needs to be.
   current surface must be closer than required.

3. No concession to flexibility.
   κ has no dependence on flexibility parameters.

4. No constraint to give port space.

ALTERNATIVE NEW METHOD

Maximize target function while keeping Ohmic power as small as possible

Target function $\quad T = T_o - (\vec{\Phi} - \overleftrightarrow{L} \cdot \vec{I})^T \cdot \overleftrightarrow{Q} \cdot (\vec{\Phi} - \overleftrightarrow{L} \cdot \vec{I})$

Power $\quad P = \vec{I}^T \cdot \overleftrightarrow{R} \cdot \vec{I}$

Resistance positive definite. Has representation

$\overleftrightarrow{R} = \overleftrightarrow{V} \cdot (\overleftrightarrow{c} \cdot \overleftrightarrow{c})^{-1} \cdot \overleftrightarrow{V}^T$ with $\overleftrightarrow{c}$ diagonal.

Let $\vec{I} = \overleftrightarrow{C} \cdot \vec{\mathcal{J}}$ with $\overleftrightarrow{C} = \overleftrightarrow{V} \cdot \overleftrightarrow{c} \cdot \overleftrightarrow{V}^T$, so $P = \vec{\mathcal{J}}^T \cdot \vec{\mathcal{J}}$.

Optimum given by $\quad \overleftrightarrow{S} \cdot \vec{\mathcal{J}} = \overleftrightarrow{L}^T \cdot \overleftrightarrow{Q} \cdot \vec{\Phi}$ with

$$\overleftrightarrow{S} \equiv \overleftrightarrow{C} \cdot \overleftrightarrow{L}^T \cdot \overleftrightarrow{Q} \cdot \overleftrightarrow{L} \cdot \overleftrightarrow{C}.$$

SVD technique gives optimum as function of s_{min} the smallest eigenvalue of $\overleftrightarrow{S}$ kept.

Problems:
1. Doesn't emphasize flexibility.
2. Sensitive to correct $\overleftrightarrow{Q}$.
3. More complicated to implement.

SUMMARY

New method given to design current potential

1. Pick N_Φ important fluxes.

   a. Each flux a set of Fourier components of $\vec{B}_o \cdot \hat{n}$ on plasma surface.

   b. Use quality matrix $\overleftrightarrow{Q}$ to do this.

2. Find most efficient coils (minimal Ohmic power) required to control the N_Φ fluxes exactly.

Advantages

1. Flexibility

   Gives independent set of currents needed to drive important fluxes independently.

2. Maximizes coil efficiency

3. Space can be reserved for ports

4. Permits greater coil/plasma separation