# DETERMINATION OF PERTURBATIONS THAT CAUSE MAGNETIC ISLANDS USING A DELTA-W CODE

Allen Boozer
Columbia University

and

Carolin Nührenberg
MPI Greifswald, Germany

# Perturbed Plasma Equilibria

$\vec{\xi} \cdot \hat{n}$ or $\vec{Q} \cdot \hat{n}$ on plasma surface define perturbed equilibria.

$\vec{\xi}(\vec{x})$ that minimizes $\delta W$ gives the perturbed equilibrium.

Resonant component of $\vec{\xi}(\vec{x})$ can jump at rational surfaces.

That jump, $[\xi]_{res}$, is proportional to the singular current.

The island half-width that would arise if no singular currents were allowed is $\delta \approx \sqrt{\frac{a}{m}[\xi]_{res}}$. (*a* is radius of surface and m is polodial mode number)

Perturbed equilibrium equations are highly singular at rational surfaces unless $dp/d\psi$ is zero there.

If $dp/d\psi$ is zero at a rational surface, the jump conditions are

(1) $\left[\vec{Q}\cdot\hat{n}\right]_{res} = 0$ from $\vec{\nabla}\cdot\vec{B} = 0$

(2) $\left[\dfrac{\partial\vec{Q}\cdot\hat{n}}{\partial\psi}\right]_{res} \propto$ singular current. $\left[\frac{\partial\vec{Q}\cdot\hat{n}}{\partial\psi}\right]_{res} \propto \left[\vec{\xi}\cdot\hat{n}\right]_{res}$

One can impose $\left[\vec{Q}\cdot\hat{n}\right]_{res} = 0$ plus one other condition at each rational surface.

The two obvious choices for second condition are:

(1) $(\vec{Q}\cdot\hat{n})_{res} = 0$ ideal MHD with singular currents.

(2) $\left[\frac{\partial\vec{Q}\cdot\hat{n}}{\partial\psi}\right]_{res} = 0$ no singular currents but $(\vec{Q}\cdot\hat{n})_{res} \neq 0$ and therefore islands.

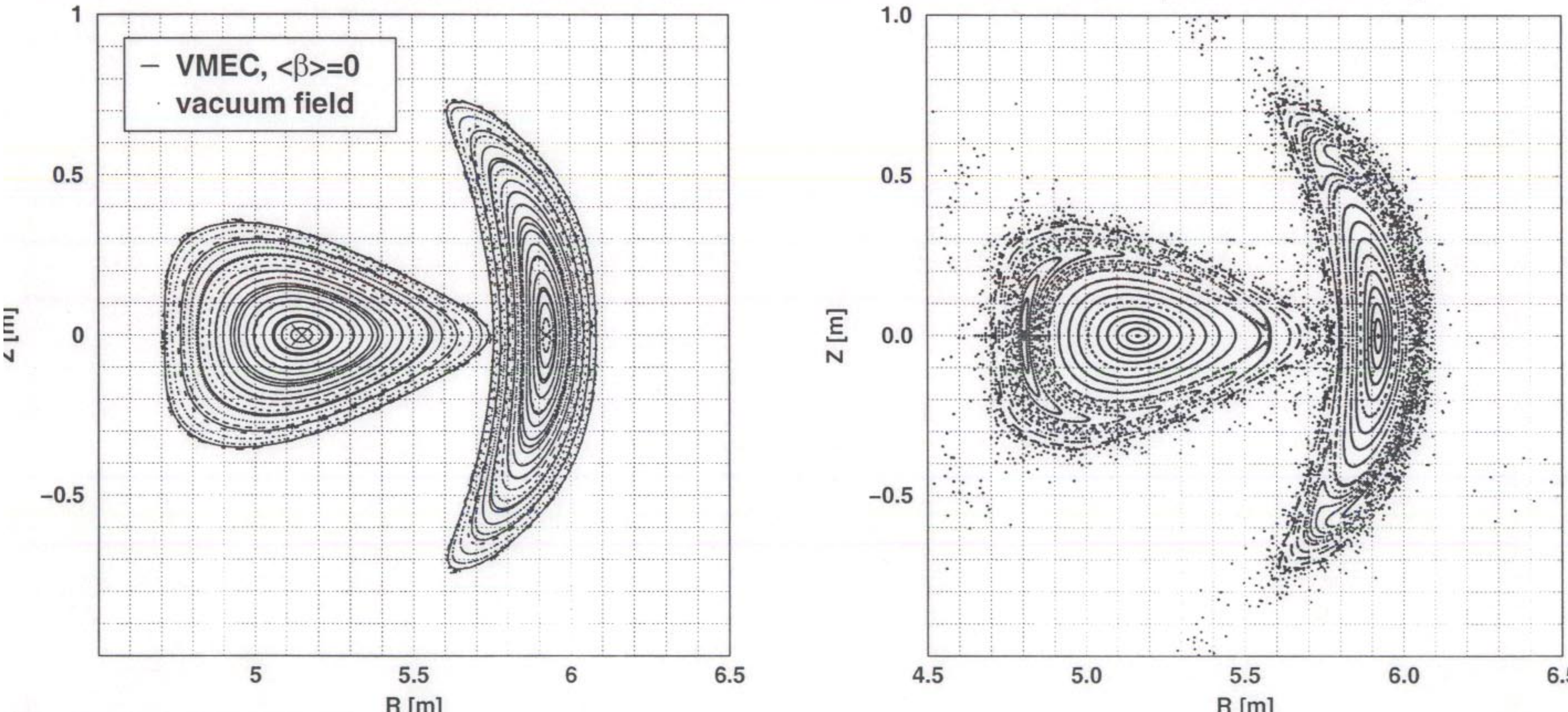
$\Delta\iota \propto \xi^2$, so equilibria without singular currents have a smooth current profile.

WENDELSTEIN 7-X

# W7-X variant with $\iota = 1$ inside coil-error induced 1/1 island

IPP

field line tracing in vacuum fields and reconstruction with VMEC (S.P. Hirshman)



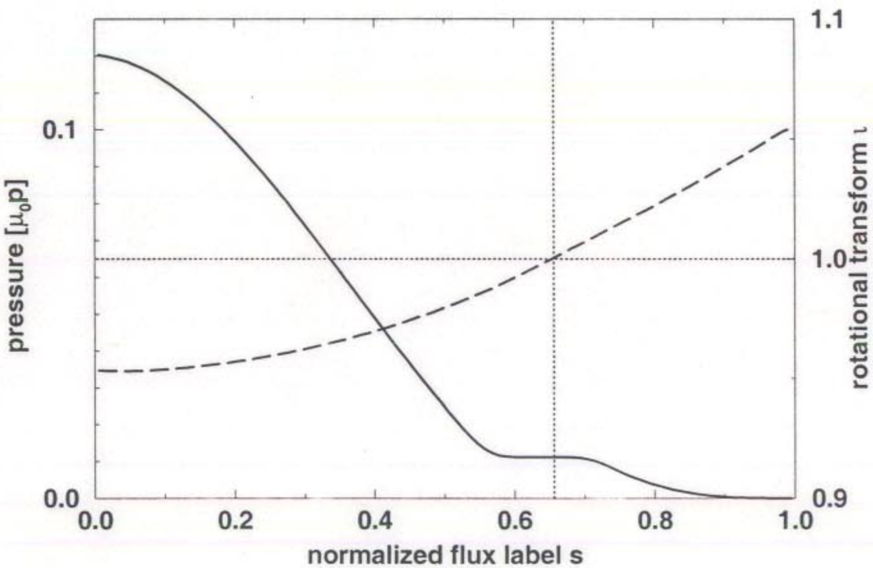

- saddle coils used to eliminate 5/5 islands
- create 1/1 island: 2 modular coils rotated around vertical axis through their geometric centers; coil peripheries displaced by 0.01 m

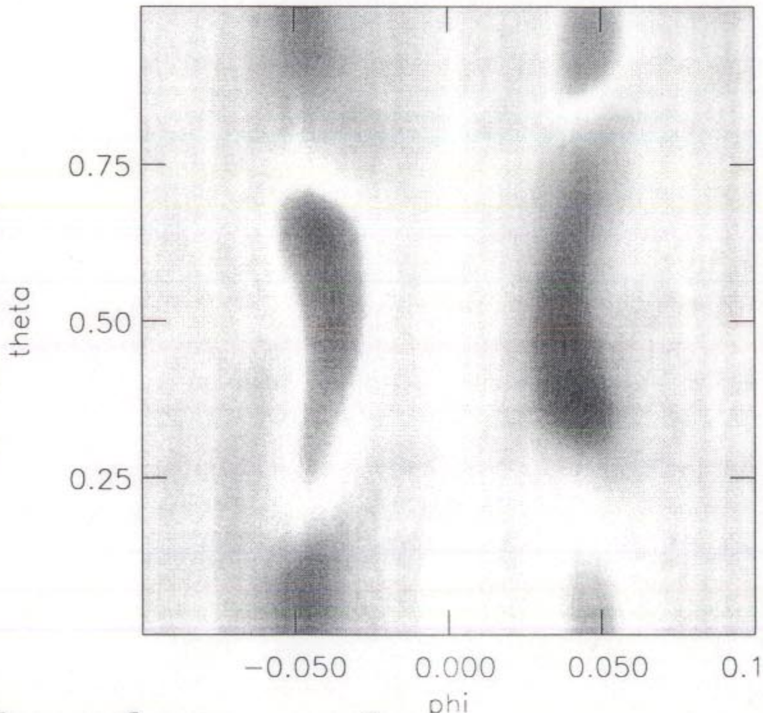
# W7-X variant with $\iota = 1$ inside

WENDELSTEIN 7-X

**profiles at $\langle\beta\rangle$ = 0.014**

**field period with displaced coils**



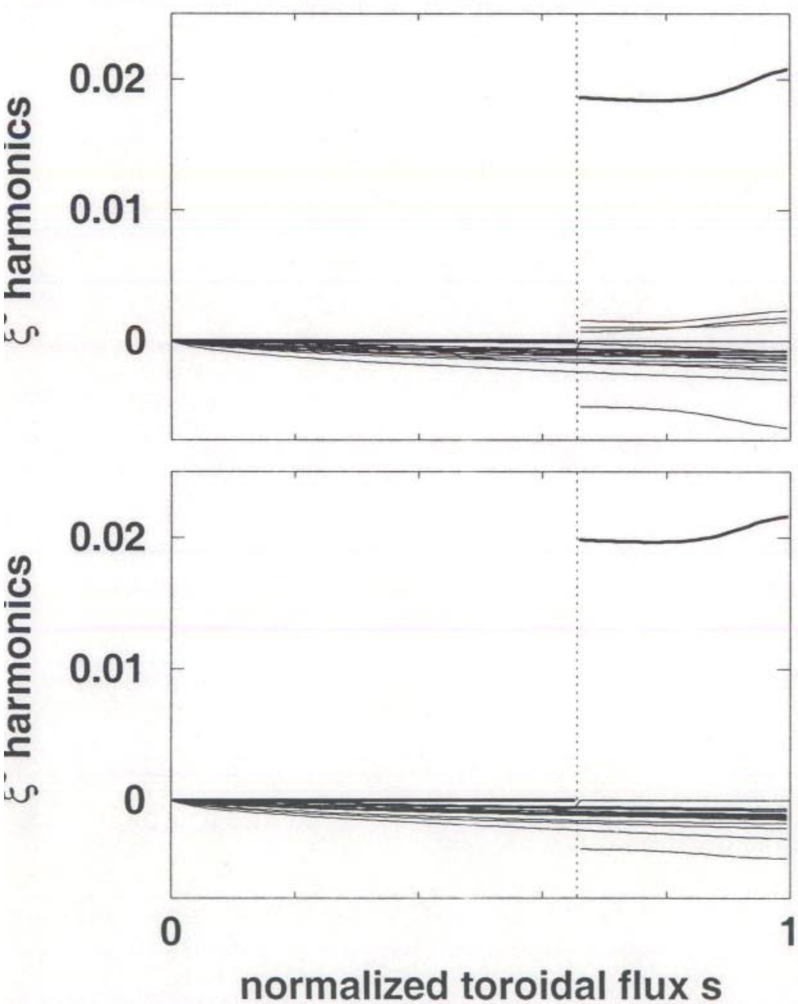




- flattened pressure near rational surface $\iota$ = 1
- use normal perturbed magnetic field on boundary of vacuum field with 1/1 island as boundary condition for $\delta W$ code
- on boundary: $B_{1,-1}^{\text{error}}$ = 0.00017 T, $B_{\text{axis}}$ = 2.5 T

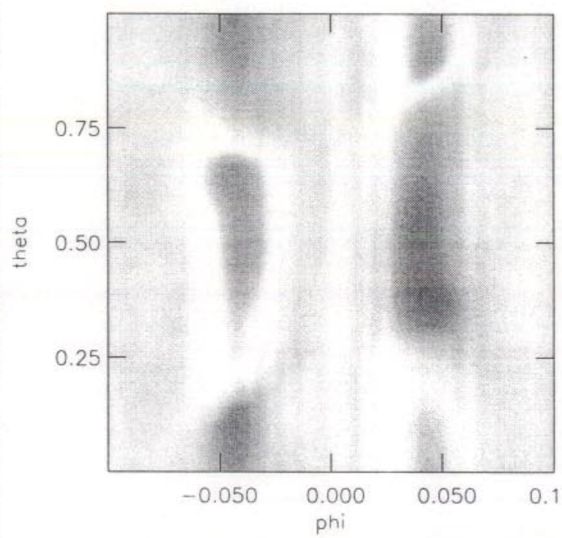
# ideal MHD for description of islands CAS3D results for W7-X variant with $\iota = 1$

WENDELSTEIN 7-X

## armonics of normal displacement




$\vec{B}_1 \cdot \vec{n}$ on rational surface
field period with displaced coils
CAS3D+BNORM
BNORM: P. Merkel




- direct solution of $\mathcal{F}\vec{\xi} = 0$ (bottom) and construction of solution from basis of eigenfunctions (top) agree very well
- on rational surface: $B^{error}_{1,-1}$=0.00018 $T$ $\Longrightarrow$ negligible error field amplification at $\langle\beta\rangle$=0.014

# Basic Theory

$$\delta W = -\tfrac{1}{2} \int_{plasma} \vec{\xi} \cdot \vec{F}(\vec{\xi}) d^3x$$

$$\vec{F} = \vec{j}_0 \times \vec{Q} + \vec{j}_1 \times \vec{B}_0 - \vec{\nabla} p_1 \qquad \vec{Q} = \vec{\nabla} \times (\vec{\xi} \times \vec{B}_0)$$

Bernstein et al showed $\delta W = \frac{1}{2\mu_0} \oint_{surface} (\vec{Q} \cdot \vec{B}_0) \vec{\xi} \cdot d\vec{a} + \delta W_p$

$$\delta W_p = \tfrac{1}{2} \int \left\{ \vec{C}^2 + \gamma p_0 (\vec{\nabla} \cdot \vec{\xi})^2 - 2(\vec{\xi} \cdot \hat{n})^2 (\vec{j}_0 \times \hat{n}) \cdot (\vec{B}_0 \cdot \vec{\nabla} \hat{n}) \right\} d^3x$$

$$\vec{C} \equiv \vec{Q} + (\vec{\xi} \cdot \hat{n})(\mu_0 \vec{j}_0 \times \hat{n})$$

Let $\quad \vec{\xi} = \xi^{\psi} \dfrac{\partial \vec{x}}{\partial \psi} + \eta \dfrac{\partial \vec{x}}{\partial \theta} + \sigma \vec{B}_{0}$

$\frac{1}{2}\int \gamma p_0 (\vec{\nabla}\cdot\vec{\xi})^2 d^3x$ minimization $\vec{B}_0 \cdot \vec{\nabla}\sigma = -\dfrac{1}{\Im}\left\{\dfrac{\partial \Im \xi^{\psi}}{\partial \psi} + \dfrac{\partial \Im \eta}{\partial \theta}\right\}$

Makes pressure constant along $\psi$–surfaces in perturbed equilibrium, $\vec{B}\cdot\vec{\nabla}p = 0$.

$\frac{1}{2}\int \vec{C}^2 d^3x$ minimization $\dfrac{\partial C_\theta}{\partial \varphi} = \dfrac{\partial C_\varphi}{\partial \theta}$ determines $\eta$.

Makes pressure constant on current surfaces of perturbed equilibrium, $\vec{j}\cdot\vec{\nabla}p = 0$.

Glasser-type representation $h(\psi,\theta,\varphi)=\sum_j f_j^*(\theta,\varphi)h_j(\psi)$

$k(\psi,\theta,\varphi)=g(\psi,\theta,\varphi)h(\psi,\theta,\varphi)$ implies $k=\vec{f}^*(\theta,\varphi)\cdot\vec{g}(\psi)\cdot\vec{h}(\psi)$

$$\vec{g}=\oint \vec{f}(\theta,\varphi)g(\psi,\theta,\varphi)\vec{f}^\dagger(\theta,\varphi)d\theta d\varphi;\ \frac{\partial\vec{f}}{\partial\theta}=-i\vec{M}\cdot\vec{f};\ \frac{\partial\vec{f}}{\partial\varphi}=i\vec{N}\cdot\vec{f}$$

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Normal field $\vec{b}=(\overleftrightarrow{N}-\iota\overleftrightarrow{M})\cdot\vec{\xi}^\psi$

Singular current at rational surface given by $\left[\vec{Q}_\theta\right]=\overleftrightarrow{M}\cdot\overleftrightarrow{\Lambda}\cdot\left[\frac{d\vec{b}}{d\psi}\right]$

Matrix $\overleftrightarrow{\Lambda}$ involves metric tensor plus $\overleftrightarrow{N}$ and $\overleftrightarrow{M}$.

## Other Applications

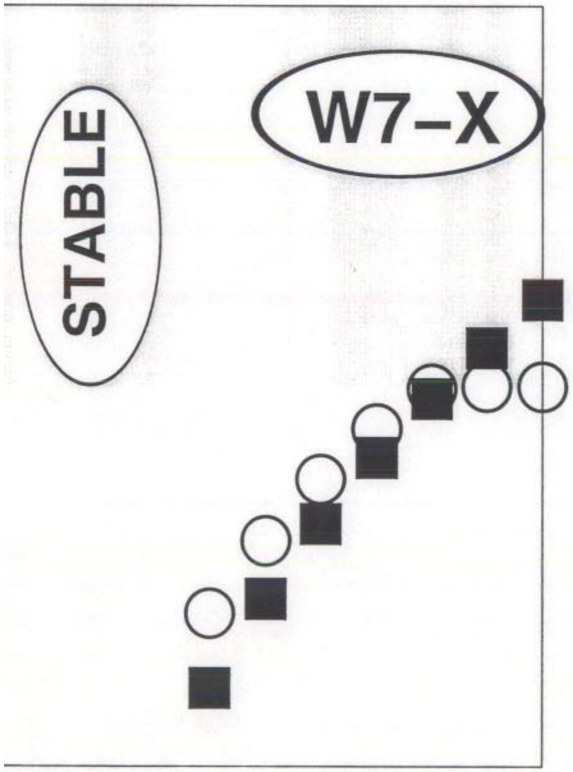
I. Stability: Plasma unstable if $\lambda \equiv \frac{\delta W}{\|\vec{\xi}\|^2} < 0$

1. Kinetic energy norm $\|\vec{\xi}\|^2 = \frac{1}{2}\int \rho \xi^2 d^3x$ conventional, but $\|\vec{\xi}\|^2 \to \infty$ & $\lambda = 0$ for stable modes with a resonant surface.

2. Magnetic energy norm $\|\vec{\xi}\|^2 = \frac{1}{2\mu_0}\int Q^2 d^3x$ allows tracking of modes from unstable to stable.

3. Wall mode norm $\|\vec{\xi}\|^2 = \frac{1}{2\mu_0} \oint_{surface} (\vec{Q}\cdot\hat{n})^2 da$

# magnetic energy normalization CAS3D results for W7-X variant




- ○ kinetic energy normalization
  - \+ physical growth rates and frequencies
  - – only indirect characterization of stable equilibria:

    stability limits via extrapolation in a series of increasingly stable cases
- ■ magnetic energy normalization
  - \+ unstable eigenvalues may be tracked into the stable region
  - \+ direct characterization of stable equilibrium with positive eigenvalue
  - – is not a physical normalization

## II. Equilibrium Correction

Suppose $\vec{\nabla} p \neq \vec{j} \times \vec{B}$ let $\vec{\nabla} p = \vec{j}_p \times \vec{B}$

$$\vec{\nabla} \times \vec{Q}_e \equiv \mu_0 (\vec{j}_p - \vec{j})$$

$\vec{Q}_e(\vec{x})$ specifies error in equilibrium

Corrected fixed boundary equilibrium given by $\vec{\xi}(\vec{x})$ that minimizes

$$\delta W = \delta W_p - \tfrac{1}{\mu_0} \int \vec{Q}_e(\vec{x}) \cdot \vec{Q} d^3 x$$