Zonal flows in stellarators

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-Zonal flows (ZFs): primarily poloidal ExB flows due to a radially-varying (m,n)=(0,0) potential $\varphi_Z(r,t)$, driven by the turbulent nonlinearities in the kinetic equation:

$$(\partial_t + \hat{H}_0)\delta f(\mathbf{z},t) = -\hat{h} f_0 - \hat{h}\delta f, \textbf{ with } \hat{A} \equiv \{,A\}.$$

-Believed important in suppressing turbulence, hence anomalous transport.

-ZF issues addressed here:

I.Shielding of ZFs.

II.Longer-time evolution of ZFs.

I.Shielding of ZFs:

-Rosenbluth & Hinton[1] (PRL, ’98) showed that in tokamaks, a (0,0) nonlinear (nl) source S(t) would be shielded by the plasma, producing a ZF amplitude $k_r^2\varphi_Z = 4\pi\delta\rho^{xt}/\boldsymbol{D}$, with external charge density perturbation $\delta\rho^{xt} \sim \int dt\, S$, and dielectric $\boldsymbol{D}(k,\omega) = (k_r \lambda_D)^{-2} g(k,\omega)$, with $g \approx k_r^2 \rho_{gi}^2 (1 + c_b q^2/\varepsilon_t^{1/2}) \approx k_r^2 \rho_{gi}^2 + F_t k_r^2 \rho_{bi}^2$, $c_b=1.6$, $F_t \approx (2/\pi)\, \varepsilon_t^{1/2}$.

-The 1st term $g^g \equiv k_r^2 \rho_{gi}^2$ comes from the gyromotion-associated “classical” polarization shielding & polarization current J^{pg}, and the 2nd term $g^b \approx F_t k_r^2 \rho_{bi}^2$ (with $\rho_{bi}^2 = \rho_{gi}^2 q^2/\varepsilon_t$) comes from an analogous bounce-associated “neoclassical” (nc) polarization shielding & current J^{pb}.

-Sugama & Watanabe[2] (PRL,’05, PoP,’06) did an analogous calculation for stellarators, finding those same 2 shielding terms, but with g^b of more complicated form, due to the extra complexity of stellarator orbits & phase space, plus an extra, drift-related term $g^d \approx F_h$, with $F_h \approx (2/\pi)\varepsilon_h^{1/2}$=fraction of helically-trapped particles (trapping index τ=h).

As in [1], their calculation is collisionless, and does expansion of kinetic eqn in $\rho_{g,b}/L$ & bounce & drift-avging to compute δf.

-Shaing[3] (PoP,’06), using “moments method” formulation of transport, computed “t-dependent viscosity” in the $1/\nu$ regime, effectively obtaining g in that higher- ν regime.

-Here, we solve the same linear-response problem as [1,2], but using the action-angle (aa) formalism (Kaufman[4], PF ('72)). This allows us to treat the complicated magnetic geometries and particle orbits of tokamaks and stellarators, and the gyro, bounce, and drift timescales $\tau_{g,b,d}$ in a more uniform manner, without having to do expansions & avging of the kinetic equation as in [1-3], and to obtain more general, transparent results for the dielectric $\boldsymbol{D} = 1 + \Sigma_s \chi_s$.

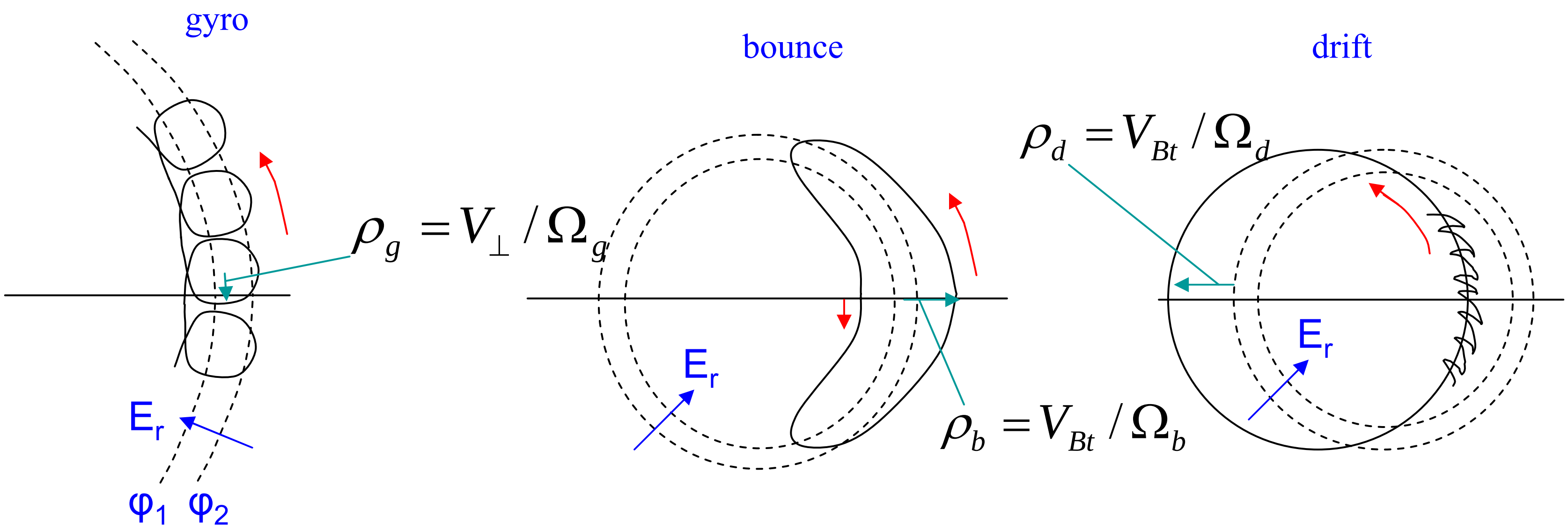
-Method recovers expressions for drift contribution g^d in [2,3], & generalizes it to wider range of physically important situations.
As noted in [2], in contrast to the gyro and bounce portions $g^{g,b}$,
electrons as well as ions can contribute to g^d

-We then use this improved description of $\boldsymbol{D}$ to study the longer-timescale, diffusive evolution of ZFs.

-Polarization shielding:

-Radial excursions on each of the 3 timescales enable particles to partially shield an external potential.

$$4\pi\delta\rho_s \approx -k^2\chi_s\delta\phi = -g_s\lambda_s^{-2}\delta\phi$$



$$\Omega_g >> \Omega_b >> \Omega_d$$

-Action-Angle (aa) Formalism:

-Reparametrize phase point **z** from more directly physical set (**r**,**p**) to (**θ**,**J**), with **J** ≡ the 3 action invariants of the unperturbed motion, and **θ** ≡ their conjugate angles.

-Collisionless (ν=0) motion governed by a Hamiltonian $H(\mathbf{z})=H_0(\mathbf{J})+h(\mathbf{z},t)$, with unperturbed & perturbing parts H_0 and $h = e\,\delta\varphi(\mathbf{r},t)$.

-In aa-variables, the particle motion is very simple:

$$\dot{\boldsymbol{\theta}} = \partial_{\mathbf{J}} H = \boldsymbol{\Omega}(\mathbf{J}) + \partial_{\mathbf{J}} h \simeq \boldsymbol{\Omega}(\mathbf{J}), \qquad (1a)$$

$$\dot{\mathbf{J}} = -\partial_{\boldsymbol{\theta}} h = -i \sum_{\mathbf{l}} \mathbf{l} h_{\mathbf{l}}(\mathbf{J},t) \exp(i\mathbf{l}\cdot\boldsymbol{\theta}), \qquad (1b)$$

$$h_{\mathbf{l}}(\mathbf{J}) \equiv (2\pi)^{-3} \oint d\boldsymbol{\theta} \exp(-i\mathbf{l}\cdot\boldsymbol{\theta}) h(\mathbf{z})\,.$$

Then the Vlasov equation may be written

$$(\partial_t + \hat{H}_0)\delta f(\mathbf{z},t) = -\delta\dot{\mathbf{J}}\cdot\partial_{\mathbf{J}} f_0 + S(\mathbf{z},t) f_0 \qquad (2)$$

with $\hat{H}_0 \equiv \{, H_0\} = \boldsymbol{\Omega}\cdot\partial_{\boldsymbol{\theta}}$ & specified source function $S(\mathbf{z},t)f_0 = -\{\delta f, h\}$

This gives $G_0^{-1}\delta f_{\mathbf{l}}(\mathbf{J},\omega) = i\mathbf{l}\cdot\partial_{\mathbf{J}} f_0 h_{\mathbf{l}}(\mathbf{J},\omega) + \delta f_{\mathbf{l}}(\mathbf{J},t=0) + S_{\mathbf{l}}(\mathbf{J},\omega) f_0$

with $G_0^{-1} \equiv (-i\omega + i\mathbf{l}\cdot\boldsymbol{\Omega} + \nu_f)$.

Then computing the density via

$$\delta\rho_s(\mathbf{x}) \;=\; \int d\mathbf{z}\rho(\mathbf{x}|\mathbf{z})\delta f_s(\mathbf{z}) \;=\; \delta\rho_{s,A+B+C}(\mathbf{x},\omega)$$

with charge density kernel $\rho(\mathbf{x}|\mathbf{z}) \equiv e_s\delta(\mathbf{x}-\mathbf{r}(\mathbf{z}))$, one finds

$$\delta\rho_{sA}(\mathbf{x},\omega) \;=\; \int d\mathbf{x}' K_s(\mathbf{x},\mathbf{x}',\omega)\delta\phi(\mathbf{x}',\omega) \tag{3a}$$

$$\delta\rho_{s,B+C}(\mathbf{x},\omega) \;=\; (2\pi)^3 \int d\mathbf{J} \sum_{\mathbf{l}} \rho_{\mathbf{l}}^*(\mathbf{x}|\mathbf{J}) G_0[\delta f_{s\mathbf{l}}(\mathbf{J},t=0) + S_{s\mathbf{l}}(\mathbf{J},\omega) f_{s0}] \tag{3b}$$

and response kernel

$$\begin{aligned} K_s(\mathbf{x},\mathbf{x}',\omega) \;&=\; (2\pi)^3 \int d\mathbf{J} \sum_{\mathbf{l}} \rho_{\mathbf{l}}^*(\mathbf{x}|\mathbf{J}) \frac{\mathbf{l}\cdot\partial_{\mathbf{J}} f_{s0}}{\mathbf{l}\cdot\mathbf{\Omega}-\omega-i\nu_f} \rho_{\mathbf{l}}(\mathbf{x}'|\mathbf{J}) \qquad (4) \\ &=\; K_s^{ad}(\mathbf{x},\mathbf{x}') + (2\pi)^3 \int d\mathbf{J} \sum_{\mathbf{l}} \rho_{\mathbf{l}}^*(\mathbf{x}|\mathbf{J}) \frac{\omega\partial_{H_0} f_{s0} + \mathbf{l}\cdot\partial_{\mathbf{J}})_{H_0} f_{s0}}{\mathbf{l}\cdot\mathbf{\Omega}-\omega-i\nu_f} \rho_{\mathbf{l}}(\mathbf{x}'|\mathbf{J}) \end{aligned}$$

with adiabatic term $K_s^{ad}(\mathbf{x},\mathbf{x}') \equiv e_s\delta(\mathbf{x}-\mathbf{x}') \int d\mathbf{z}\rho(\mathbf{x}|\mathbf{z})\partial_{H_0} f_{s0}.$

-For a local Maxwellian form $f_M(\mathbf{J}) \equiv \dfrac{n_0}{(2\pi MT)^{3/2}} \exp[-(H_0 - e\Phi_a)/T)$ (5)

one finds $K_s^{ad}(\mathbf{x},\mathbf{x}') = -1/(4\pi\lambda_s^2(\mathbf{x}))\delta(\mathbf{x}-\mathbf{x}')$

-Example: Slab geometry:

-Magnetic field: $\mathbf{B} = \hat{z}\partial_x\psi = \nabla\psi \times \nabla y$, $\psi(x) \equiv A_y(x)$

-Hamiltonian: $H_0(\mathbf{r},\mathbf{p}) = (p_x^2 + p_z^2)/(2M) + (p_y - \frac{e}{c}A_y(x))^2/(2M) + e\Phi(x)$

-Transform from (r,p) to aa variables:

$\boldsymbol{\theta} = (\theta_g, \bar{y}, z)$, $\mathbf{J} = (J_g \equiv (Mc/e)\mu, p_y = \frac{e}{c}\bar{\psi} \equiv \frac{e}{c}\psi(\bar{x}), p_z)$

-Using eikonal form for mode structure,

$\phi_a(\mathbf{x}) = \bar{\phi}_a(x) \exp i\eta_a(\mathbf{x})$, with $\eta_a(\mathbf{x}) \equiv [\int^x dx'\, k_x(x') + k_y y + k_z z]$, find

$\int d\mathbf{x}\phi_a^*(\mathbf{x})\delta\rho_{sA}(\mathbf{x}) = -\int dx V'|\bar{\phi}_a|^2(x)\frac{k^2}{4\pi}\chi_s(\mathbf{k},\omega)$, with susceptibility

$\chi_s(\mathbf{k},\omega) = (k\lambda_s)^{-2}g_s(\mathbf{k},\omega))$, $g_s(\mathbf{k},\omega) = 1 - \sum_{l_g}\langle J_{l_g}^2(z_g)\frac{\omega - \omega_{*s}^f}{\omega - \mathbf{l}\cdot\mathbf{\Omega} + i\nu_{fs}}\rangle$,

where $z_g \equiv k_\perp\rho_g$ and $\mathbf{l}\cdot\mathbf{\Omega} = l_g\Omega_g + k_y\dot{\bar{y}} + k_z v_z$.

-Toroidal geometry: Magnetic field given by:

$$\mathbf{B} = \nabla\psi\times\nabla\theta + \nabla\zeta\times\nabla\psi_p = \nabla\psi\times\nabla\alpha_p \qquad \alpha_p \equiv \theta - \iota\zeta.$$

$$B(\mathbf{x}) = \bar{B}(r)[1-\epsilon_t(r)\cos\theta - \delta_h(\mathbf{x})\cos\eta_0] \qquad \epsilon_h(r) \equiv \langle\delta_h\rangle,\ \eta_0 \equiv n_0\zeta - m_0\theta. \quad (6)$$

-Specialize aa variables to

$$\boldsymbol{\theta} = (\theta_g, \theta_b, \theta_d \simeq \bar{\alpha}_p),\ \mathbf{J} = (J_g \equiv (Mc/e)\mu, J_b, J_d \simeq (e/c)\bar{\psi})$$

-Using eikonal form for mode structure,

$$\phi_a(\mathbf{x}) = \bar{\phi}_a(r)\exp i\eta_a(\mathbf{x}) \qquad \eta_a(\mathbf{x}) \equiv [\textstyle\int^r dr'\,k_r(r') + m\theta + n\zeta]$$

one obtains radially-local response equation:

$$k^2\mathcal{D}(\mathbf{k},\omega)\frac{e_i\bar{\phi}_a(r)}{T_i} = \sum_s \lambda_{si}^{-2}\sum_{\mathbf{l}}\langle G^*_{\mathbf{l}a}(\mathbf{J})\frac{i[\delta f_{s\mathbf{l}}(t=0)/f_{s0} + S_{s\mathbf{l}}(\omega)]}{(\omega - \mathbf{l}\cdot\mathbf{\Omega} + i\nu_{fs})}\rangle \quad (7)$$

with $\mathbf{l}\cdot\mathbf{\Omega} \equiv l_g\Omega_g + l_b\Omega_b + l_d\Omega_d$,

$$\mathcal{D}(\mathbf{k},\omega) \equiv 1 + \textstyle\sum_s \chi_s(\mathbf{k},\omega),\ \chi_s(\mathbf{k},\omega) = (k\lambda_s)^{-2}g_s(\mathbf{k},\omega)) \quad (8a)$$

$$g_s(\mathbf{k},\omega) = 1 - \sum_{\mathbf{l}}\langle|G_{\mathbf{l}a}(\mathbf{J})|^2\frac{\omega - \omega^f_{*s}}{\omega - \mathbf{l}\cdot\mathbf{\Omega} + i\nu_{fs}}\rangle \quad (8b)$$

Using orbit description

$$r - r_d = \delta r^{(d)}(\theta_d) + \delta r^{(b)}(\theta_b) + \delta r^{(g)}(\theta_g) \simeq \rho_d\cos\theta_d + \rho_b\cos\theta_b + \rho_g\cos\theta_g \quad (9a)$$

obtain orbit-avging factor

$$G_{\mathbf{l}a}(\mathbf{J}) \equiv \oint\frac{d\boldsymbol{\theta}}{(2\pi)^3}e^{-i\mathbf{l}\cdot\boldsymbol{\theta}}e^{i\eta_a(\mathbf{r})} = J_{l_g}(z_g)J_{l_b}(z_b)J_{l_d}(z_d)e^{-i\xi_a}, \text{ with } z_{g,b,d} \equiv k_r\rho_{g,b,d} \quad (9b)$$

-Have $z_{g,b} \ll 1, \Rightarrow \sum_{\mathbf{l}} \to \sum_{l_d}$ in $g_s(k,\omega)$.
-For $\omega >> \Omega_d$, integrand in (8b) about constant over l$_d$-range $\Delta l_d \sim z_d$ over which integrand appreciable, so one can do summation, using $\Sigma_l J_l^2(z) = 1$
-For $\omega << \Omega_d$, sum dominated by $l_d = 0$ term. Thus, have limiting forms

$$g_s(\mathbf{k},\omega) \simeq 1 - \Lambda_{0b}(b_g, b_b), (\omega \gg \Omega_d),$$
$$g_s(\mathbf{k},\omega) \simeq 1 - \Lambda_{0d}(b_g, b_b, b_d), (\omega \ll \Omega_d). \qquad (10)$$

where $\Lambda_{0d}(b_g, b_b, b_d) \equiv \langle J_g^2 J_b^2 J_d^2 \rangle, \Lambda_{0b}(b_g, b_b) \equiv \Lambda_{0d}(b_g, b_b, b_d = 0) \equiv \langle J_g^2 J_b^2 \rangle,$

$$\Lambda_0(b_g) \equiv \Lambda_{0b}(b_g, b_b = 0) \equiv \langle J_g^2 \rangle = I_0(b_g) e^{-b_g},$$

$$J_{g,b,d}^2 \equiv J_0^2(z_{g,b,d}), b_g \equiv k_r^2 \rho_{gT}^2, b_b = b_g q^2/(F_t \epsilon_t^{1/2}), b_d \equiv k_r^2 \rho_{dT}^2, \rho_{gT} \equiv v_T/\Omega_g$$

-Approximately evaluate Λ_{0d} using expansion $J_0(z) \approx 1 - (z/2)^2$:

$$\Lambda_{0d}(b_g, b_b, b_d) \simeq 1 - \tfrac{1}{2}\langle z_g^2 \rangle - \tfrac{1}{2}\langle z_b^2 \rangle - \tfrac{1}{2}\langle z_d^2 \rangle = 1 - b_g - F_t c_b b_b - F_h c_d b_d,$$

with $c_b \simeq 3\sqrt{2}/\pi \simeq 1.4$, $c_d \simeq (15/2)$, $F_h = (2/\pi)\sqrt{2\epsilon_h}$

-Thus,

$$g_s(\mathbf{k},\omega) \simeq b_g + F_t c_b b_b = g_s^g + g_s^b, (\omega \gg \Omega_d), \qquad (11)$$

$$g_s(\mathbf{k},\omega) \simeq b_g + F_t c_b b_b + F_h c_d b_d = g_s^g + g_s^b + g_s^d, (\omega \ll \Omega_d).$$

-Notes on these results:

-While $g_e^{g,b} << g_i^{g,b}$, can have $g_e^d \sim g_i^d$, because
while $\rho_{g,b}^e << \rho_{g,b}^i$, have $\rho_d^e \sim \rho_d^i$.

-For $\nu_f \geq \Omega_{d,}$ (eg, in $1/\nu$ –regime), successive l_d-peaks broaden until dominant $\Delta l_d \approx z_d$ harmonics overlap, and again lose drift-avging contribution g^d to g.

-Goal of nc transport optimization is basically to reduce F_h or $\rho_d \approx v_{Bt}/\Omega_d$, either by decreasing v_{Bt} (eg, with QS designs), or by enhancing Ω_d (eg, by operating at electron root). See from $g^d \approx F_h (k_r \rho_d)^2$ that this is also has the effect of reducing the drift-shielding g^d , hence of enhancing the ZFs.

-Note correspondence between each polarization-shielding mechanism, and A corresponding 'branch' of collisional transport:

Transport mech j:	D^j	Polarization shielding	g^j
classical transport	D^g	gyro (classical) shielding	g^g
axisym nc “	D^{bt}	bounce-nc shielding	g^{bt}
helically-sym nc “	D^{bh}	bounce-nc shielding	g^{bh}
superbanana “	D^{dh}	drift-nc shielding	g^{bh}
banana-drift “	D^{dt}	drift-nc shielding	g^{bt}

$$D^j \approx F_j \nu_f (\Delta r_j)^2 , \qquad g^j \approx F_j (k_r \Delta r_j)^2 \qquad (12)$$

$$\Rightarrow g^{j'}/g^j \approx (D^{j'}/D^j)(\nu_{fj}/\nu_{fj'})$$

-Rel’n to Sugama & Watanabe[2] calculation:

They obtain $g^d \approx F_h$, instead of $g^d \approx F_h (k_r \rho_d)^2$ above.

-Reason: Ordering or the kinetic eqn in [1,2] neglects $\Omega_d \partial_{\theta_d} \delta f$. Thus, they are working in the limit $\rho_d \approx V_{Bt} / \Omega_d \to \infty$. In that $z_d \to \infty$ limit, above form for Λ_{0d} recovers this:

$$\Lambda_{0d} \approx (1-b_g)[F_p(1-b_{bp}) + F_t(1-b_b) + F_h \langle J_d^2 \rangle_h] \tag{13}$$

$$\xrightarrow{z_d \to 0} (F_p + F_t + F_h) - b_g - F_t b_b - F_h b_d$$

$$\xrightarrow{z_d \to \infty} (F_p + F_t) - b_g - F_t b_b$$

-Rel’n to Shaing[3] “t-dependent viscosity” calculation:

Uses moment-method formulation of transport, where $\Gamma^s \sim \langle B_t \cdot \nabla \cdot \vec{\vec{\pi}} \rangle$.

Orders bounce-avged kinetic eqn $\partial_t \delta f \simeq -\bar{\dot{r}} \partial_r f_0 + C \delta f,$

in 1/ν and banana regimes in (1) (2) (3) (14)

“zero-frequency” limit (neglect (1)): Compute $\Gamma^s, \vec{\vec{\pi}}$ from standard soln $\delta f_{1/\nu} \approx \bar{\dot{r}} \partial_r f_0 / \nu_h$ of (14), and

“high-frequency” limit (neglect (3)): $\delta f_t \approx \bar{\dot{r}} \partial_r f_0 / \gamma$, where $\partial_t \delta f \to \gamma \delta f$

This result can be recovered from (l_g=0 , l_b=0, l_d=±1) limit of aa sol’n

$\delta f = \sum_{l_d} \delta f_{l_d} \exp(i l_d \theta_d)$, taking ω→iγ : $\delta f \simeq \frac{1}{2} v_{Bt} \partial_r f_0 / [\Omega_d - i(\gamma + \nu_h)] e^{i\theta_d} + c.c.$

-Longer (diffusive) timescale ZF evolution:

-From surface-averaging Ampere's law, one has $\partial_t E_r = -4\pi J_r$ (13a)

where $E_r \equiv \left\langle \nabla r \cdot \vec{E} \right\rangle$, and

$$J_r \equiv \left\langle \nabla r \cdot \vec{J} \right\rangle = (4\pi)^{-1} \chi \partial_t E_r + \sigma (E_r - E_a) + F_S / B \qquad (13b)$$

Here, $F_S \equiv$ force exerted by turbulence normal to B in a surface (assumed random), $\sigma \equiv \partial_{Er}$(nonambipolar particle transport flux), and $\chi \equiv$ dielectric shielding (as before).

Putting (1b) into (1a), one has a Langevin-like equation (here in ω domain):

$$-i\omega E(\omega) + \gamma_E(\omega) E(\omega) = c_S(\omega), \qquad E \equiv E_r - E_a \qquad (14)$$

where $\gamma_E(\omega) \equiv 4\pi\sigma/\boldsymbol{D}(\omega)$, $c_S(\omega) \equiv - 4\pi F_S/B\, \boldsymbol{D}$, and $\boldsymbol{D} \equiv 1 + \chi$.

-For $\boldsymbol{D}(\omega) = \boldsymbol{D}_0$ ω-independent, becomes standard Langevin eqn

$$\partial_t E(t) + \gamma_E E(t) = c_S(t), \qquad (15)$$

restoring term

Induces diffusion, with diff.coef

$$D_E \equiv \int_0^\infty d\tau < c_s(t) c_s(t-\tau) >_p \qquad (16)$$

-Probability distribution function p satifies

$$\partial_t p(t) = \partial_E (D_E \partial_E p + \gamma_E E p), \qquad (17)$$

resulting in

$$\partial_t < E >_p = -\gamma_E < E >_p \qquad (18)$$

$$\frac{1}{2} \partial_t < E^2 >_p = D_E - \gamma_E < E^2 >_p . \qquad (19)$$

-See here the balance between diffusion & the restoring toward $E_r = E_a$.

-In steady state, these give

$$p(E) = p_0 \exp(-\gamma_E E^2 / 2D_E), \quad \text{and} \quad < E^2 >_p = D_E / \gamma_E . \qquad (20)$$

-Summary:

I.Shielding of ZFs:

-Have used aa-formalism to obtain succinct, generalized expressions for polarization shielding function $g=g^g+g^b+g^d$, valid for arbitrary $\rho_{g,b,d}$, or $z_{g,b,d}$. Recovers results of previous work[2,3] in appropriate limits.

-In same limit z_d<<1 taken for $z_{g,b}$, form for $g^d \approx F_h (k_r \Delta r_d)^2$ analogous to those for $g^{g,b}$.

-Each shielding mechanism [$g^j \approx F_j (k_r \Delta r_j)^2$] corresponds to a collisional transport mechanism: $D^j \approx F_j \nu_f (\Delta r_j)^2$.

-Thus, as suggested in earlier work, neoclassically-optimized stellarators should have less damping of ZFs, tending to also diminish turbulent transport.

II.Longer-time evolution of ZFs:

-Governed by a Langevin-like equation for $E \equiv E_r - E_a$, resulting in evolution equation for the pdf for E $\partial_t p(t) = \partial_E (D_E \partial_E p + \gamma_E E p),$

with diffusion coefficient $D_E \sim 1/g^2$ & restoring force $\gamma_E \sim 1/g$

Making $< E^2 >_p = D_E / \gamma_E \sim 1/g$ decrease with increasing g.

END