Finite Larmor Radius Stabilization of Ideal Ballooning Instabilities in 3-D plasmas

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Motivation

- The nature of ideal MHD ballooning modes in 3-D systems differs qualitatively from ballooning modes in 2-D systems
  - Field-line dependence of ballooning mode eigenvalues
  - This typically corresponds to a global mode that is highly localized on the magnetic surface ~ Can nonideal physics (e. g. FLR physics) more easily stabilize these localized modes in 3-D relative to 2-D systems?
  - This work, include FLR effects in ballooning mode formalism of 3-D systems

Ideal MHD ordering and WKB-like formalism is used throughout

- For ideal MHD ballooning modes

$$(\omega^2 \rho \overleftrightarrow{I} + \overleftrightarrow{F}) \cdot \vec{\xi} = 0$$

- Use large $\mathbf{k}_\perp$ expansion

$$\xi(\vec{x}) = \hat{\xi}(\vec{x}) e^{i\frac{S(\vec{x})}{\varepsilon}}$$

$$\vec{B} \cdot \frac{\nabla S}{\varepsilon} = \vec{B} \cdot k_\perp = 0$$

- $1/\varepsilon \sim n$ ("infinite-n theory") large toroidal mode number
- Leading order solution leads to an ordinary differential equation for ξ^ψ along the field line, the ballooning equation

Ballooning equation

- Equation of motion to order $\mathcal{O}(\epsilon^{-1})$, "ballooning equation"

$$(\mathbf{B}\cdot\nabla)\,\frac{|\mathbf{k}_\perp|^2}{B^2}\,(\mathbf{B}\cdot\nabla)\,\hat{\xi} - \frac{\mathbf{B}\times\kappa}{B^2}\cdot\mathbf{k}_\perp\frac{\mathbf{B}\times\nabla p_0}{B^2}\cdot\mathbf{k}_\perp\hat{\xi} = -\frac{|\mathbf{k}_\perp|^2}{v_A^2}\omega^2\hat{\xi}$$

- Solved along each field line for all $\mathbf{k}_\perp$ to find "most unstable" field line and orientation

$$\omega^2 = \lambda(\alpha, q, \theta_k)$$

where

$$\begin{aligned}
\alpha &\equiv \text{field line label} \\
q &\equiv \text{surface label} \\
\theta_k &\equiv \frac{k_q}{k_\alpha} \\
\mathbf{k}_\perp &= k_q\nabla q + k_\alpha\nabla\alpha \qquad k_q = \frac{\partial S}{\partial q} \qquad k_\alpha = \frac{\partial S}{\partial \alpha}
\end{aligned}$$

Two-fluid physics brings in finite Larmor radius effects

- MHD equations modified by Hall-MHD terms in Ohm's law and gyroviscosity

$$\vec{E} + \vec{v} \times \vec{B} = \frac{1}{ne}(\vec{J} \times \vec{B} - \nabla p_e)$$

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}\right) = \vec{J} \times \vec{B} - \nabla p - \nabla \cdot \Pi_i^{gv}$$

- Order such that FLR corrections enter

$$k_\perp \rho_i \sim \frac{\omega_{*i}}{\omega} \sim O(1),$$

$$\omega_{*i} = \vec{k} \cdot \vec{v}_{di} = \vec{k} \cdot \frac{\vec{B} \times \nabla p_i}{neB^2} = k_\alpha \Omega_{*i}$$

- Modified ballooning equation

$$(\vec{B} \cdot \nabla) \frac{|k_\perp|^2}{B^2} (\vec{B} \cdot \nabla) \hat{\xi} - \frac{\vec{B} \times \vec{\kappa}}{B^2} \cdot \vec{k}_\perp \frac{\vec{B} \times \nabla p}{B^2} \cdot \vec{k}_\perp \hat{\xi}$$

$$= -\frac{|k_\perp|^2}{B^2} \omega(\omega - k_\alpha \Omega_{*i}) \hat{\xi}$$

Ideal and non-ideal equations

- Differ only in right hand sides

$$
\begin{aligned}
(\mathbf{B}\cdot\nabla)\frac{|\mathbf{k}_\perp|^2}{B^2}(\mathbf{B}\cdot\nabla)\hat{\xi} - \frac{\mathbf{B}\times\kappa}{B^2}\cdot\mathbf{k}_\perp\frac{\mathbf{B}\times\nabla p_0}{B^2}\cdot\mathbf{k}_\perp\hat{\xi} &= -\frac{|\mathbf{k}_\perp|^2}{v_A^2}\omega^2\hat{\xi} \\
(\mathbf{B}\cdot\nabla)\frac{|\mathbf{k}_\perp|^2}{B^2}(\mathbf{B}\cdot\nabla)\hat{\xi} - \frac{\mathbf{B}\times\kappa}{B^2}\cdot\mathbf{k}_\perp\frac{\mathbf{B}\times\nabla p_0}{B^2}\cdot\mathbf{k}_\perp\hat{\xi} &= -\frac{|\mathbf{k}_\perp|^2}{v_A^2}\omega(\omega - k_\alpha\Omega_{*i})\hat{\xi}
\end{aligned}
$$

- Define

$$\Omega^2 = \omega(\omega - k_\alpha\Omega_{*i})$$

so BE *eigenvalue problem* same for ideal and non-ideal cases

$$
\begin{aligned}
\omega^2 &= \lambda(\alpha, q, \theta_k) \\
\Omega^2 &= \lambda(\alpha, q, \theta_k)
\end{aligned}
$$

Semi-classical quantization

- Not all $\omega^2 = \lambda(\alpha, q, \theta_k)$ correspond to a "quantizable mode"
- To quantize apply "semi-classical methods" (i.e., classical methods used in solving Schrödinger equation) to ballooning modes [Dewar and Glasser, *Phys. Fluids* **26**(10), p. 3038 (1983).]

  $\Rightarrow$ trace rays of constant ω^2

  $\Rightarrow$ compute action integrals

  $\Rightarrow$ designate values of ω^2 that obey physical quantization rules as "modes"

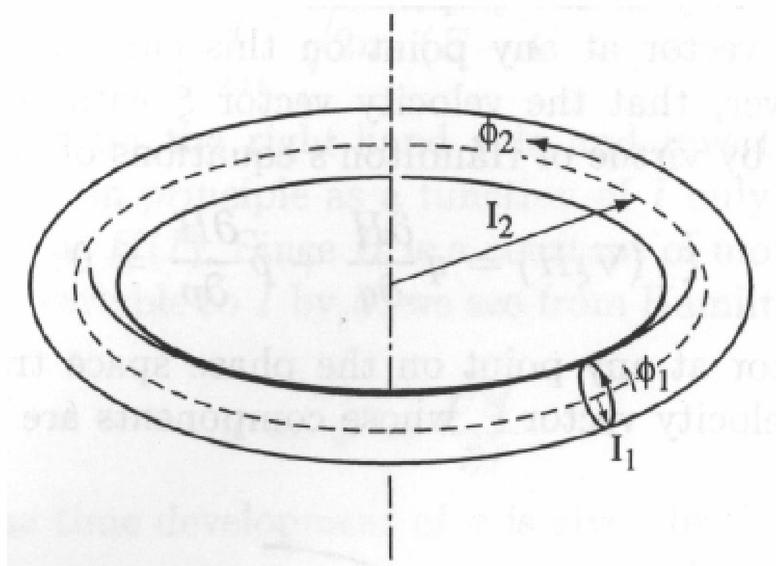
Rays of constant ω^2

- Rays of constant $\omega^2 = \lambda(\alpha, q, k_\alpha, k_q)$ obey

$$\dot{\alpha} = \frac{\partial\lambda}{\partial k_\alpha} \qquad \dot{k}_\alpha = -\frac{\partial\lambda}{\partial\alpha}$$

$$\dot{q} = \frac{\partial\lambda}{\partial k_q} \qquad \dot{k}_\alpha = -\frac{\partial\lambda}{\partial q}$$

- If system is *integrable*, phase space has torus structure

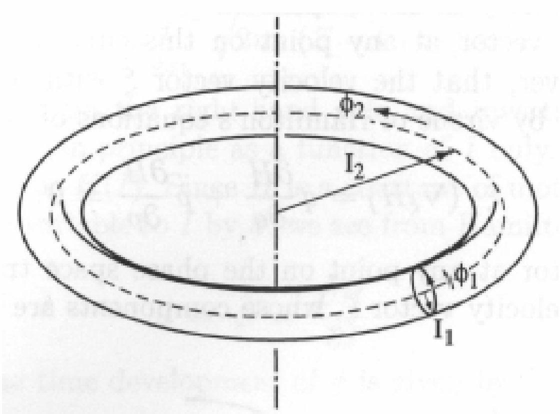


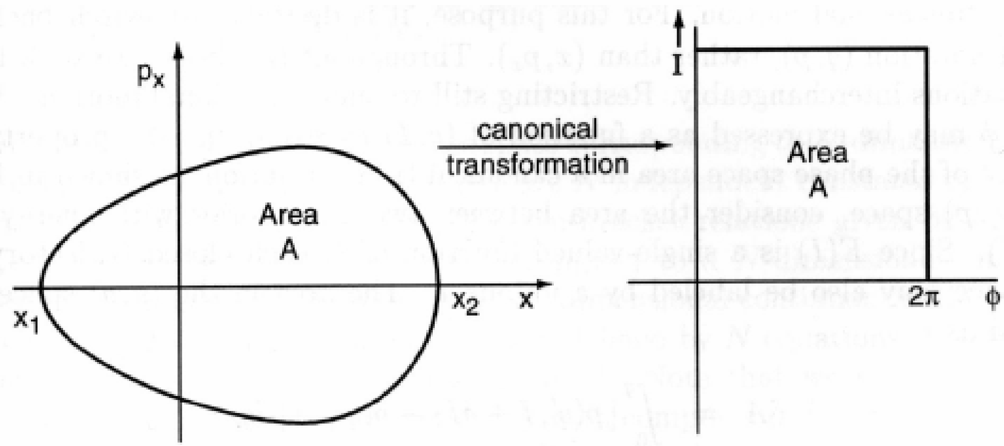
Action

- Let $\mathbf{q} = (\alpha, q)$ and $\mathbf{p} = (k_\alpha, k_q)$
- Chose a candidate "mode" by picking ω^2 and $(\mathbf{q}_0, \mathbf{p}_0)$, and consider the "action"

$$S = \int_{\mathbf{q}_0}^{\mathbf{q}} \mathbf{p} \cdot \mathbf{dq} = \int_{\mathbf{q}_0}^{\mathbf{q}} \nabla S \cdot \mathbf{dq}$$

around the α and q contours





Modes correspond to quantizable action integrals

- Action integrals of WKB trajectories are quantized

$$\frac{1}{\varepsilon}\oint_{\alpha} k_{\alpha} d\alpha = (2n_{\alpha}+1)\pi$$

$$\frac{1}{\varepsilon}\oint_{\alpha} k_{q} dq = (2n_{q}+1)\pi$$

- Quantizable trajectories are actual MHD modes of the system.

The inclusion of FLR physics in 2-D systems is straightforward

- In tokamaks, the $I_\alpha = \int k_\alpha \, d\alpha$ quantization is trivial --- toroidal mode number n is a good quantum number. Local eigenvalues are independent of field line label, α ″ k_α is conserved along ray trajectories.
  - $\omega_i^* = k_\alpha (dp/d\psi)/ne = k_\alpha \Omega_{*i}$ is constant on WKB orbit equations. Hence, $\omega^2 = \lambda$ is conserved on WKB orbits and the frequency satisfies (Tang et al, 1980)

$$\omega = \frac{k_\alpha \Omega_{*i}}{2} \pm \sqrt{\frac{k_\alpha^2 \Omega_{*i}^2 + 4\lambda}{2}}$$

  - For unstable local eigenvalue $\lambda < 0$, stability is obtained if the criterion is satisfied

$$k_\alpha^2 \Omega_{*i}^2 + 4\lambda > 0$$

In 3-D systems, the inclusion of FLR physics introduces complications

- In stellarators, local eigenvalues are generally functions of field lines, $\lambda = \lambda(\psi, \theta_k, \alpha)$ --- k_α and λ are no longer constants on WKB rays. (Nevins and Pearlstein, ‘88)
- Only the α ray equation changes,

$$\dot{\alpha} = \frac{\partial \lambda}{\partial k_\alpha} + \omega \Omega_{*i}$$

- Given unstable mode ($\lambda < 0$) described by particular values (α_o, q_o, $k_{\alpha o}$, k_{qo}), if

$$k_\alpha^2 \Omega_{*i}^2 + 4\lambda > 0$$

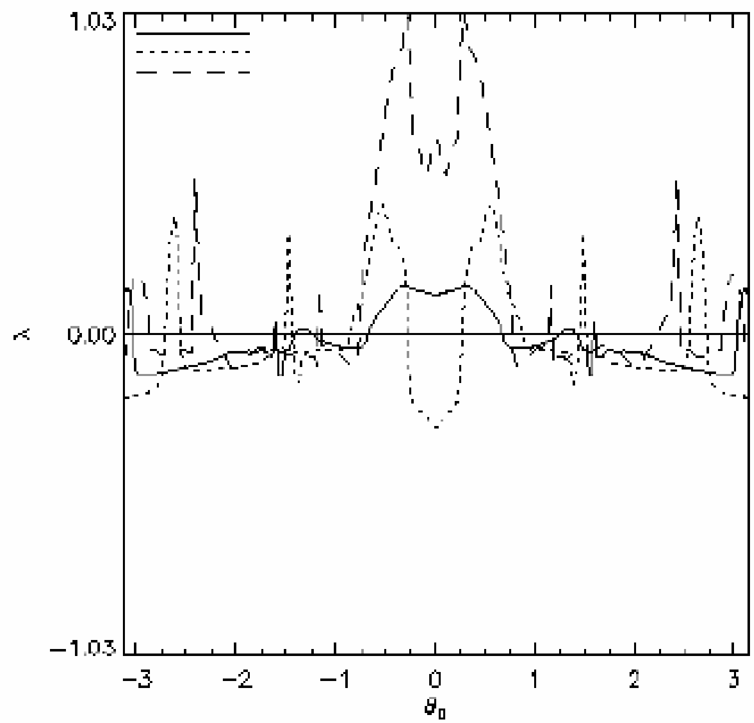
Mode is stabilized

3-d toy model

- Pick a "toy λ" to emulate what is seen in stellarator ballooning eigenvalue calculations (Hudson and Hegna PoP submitted)

  $\Rightarrow$ fast α dependence

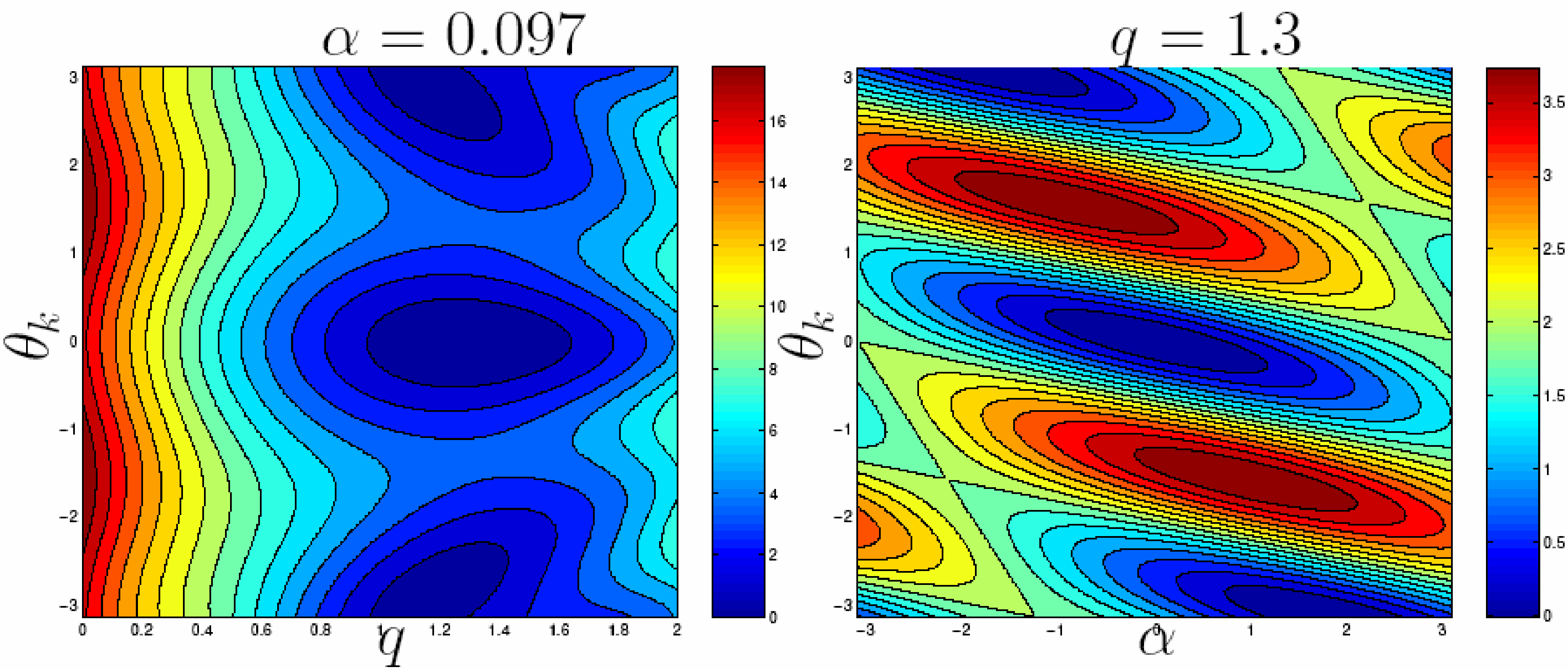
  $\Rightarrow$ θ_0 line label, traces for different surfaces



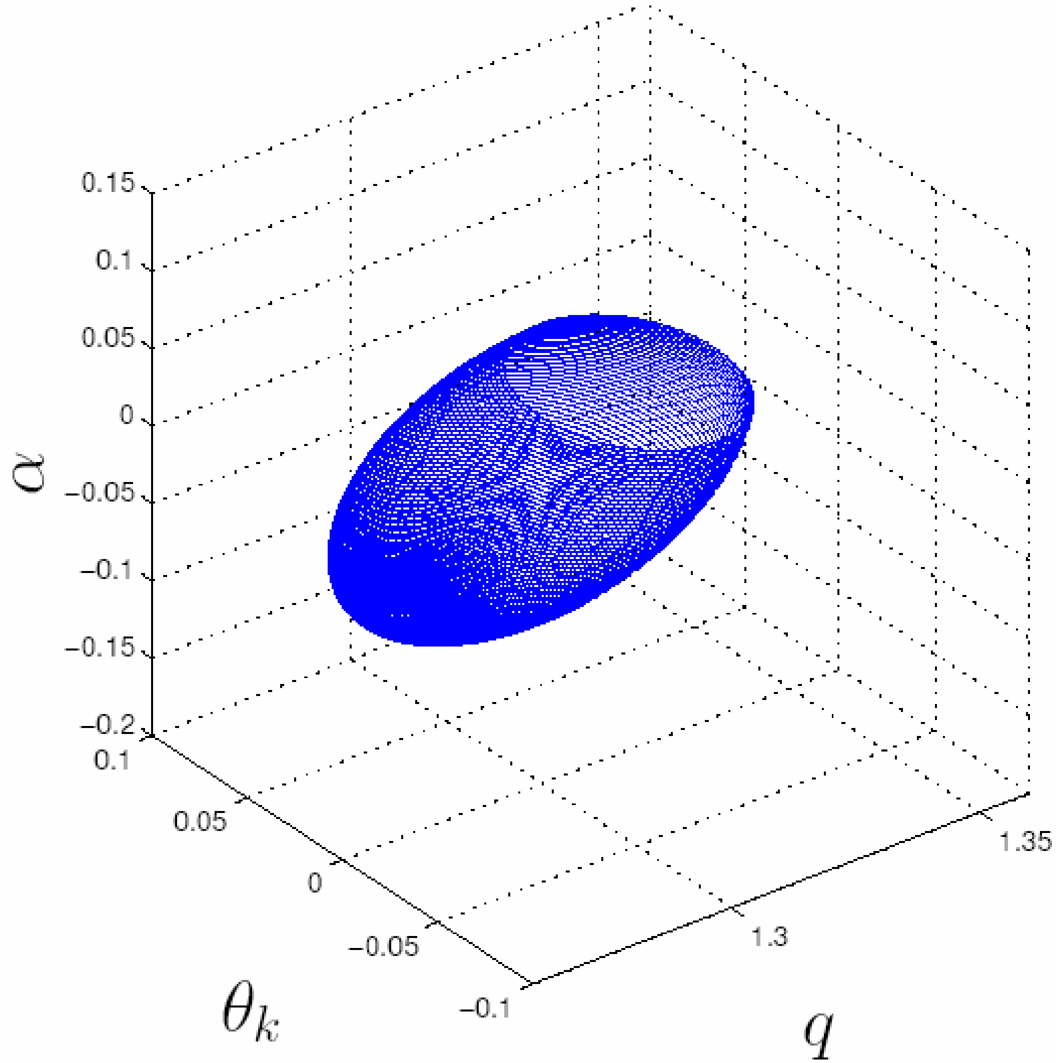
$\lambda(\alpha, q, \theta_k)$

- λ must be periodic in θ_k and $\alpha + q\theta_k$

$$\lambda(\alpha, q, \theta_k) \;=\; A + B(q - q_0)^2 + C\cos m\theta_k + D\cos n(\alpha + q\theta_k)$$



Ideal ray orbits lie on topological spheroids in phase space labeled by (q,θ_k,α)



FLR "stabilization"

- Ray equations for constant ω

$$\dot{\alpha} = \frac{\partial\lambda}{\partial k_\alpha} + \omega\Omega_{*i} \qquad \dot{k}_\alpha = -\frac{\partial\lambda}{\partial\alpha}$$

$$\dot{q} = \frac{\partial\lambda}{\partial k_q} \qquad \dot{k}_\alpha = -\frac{\partial\lambda}{\partial q}$$

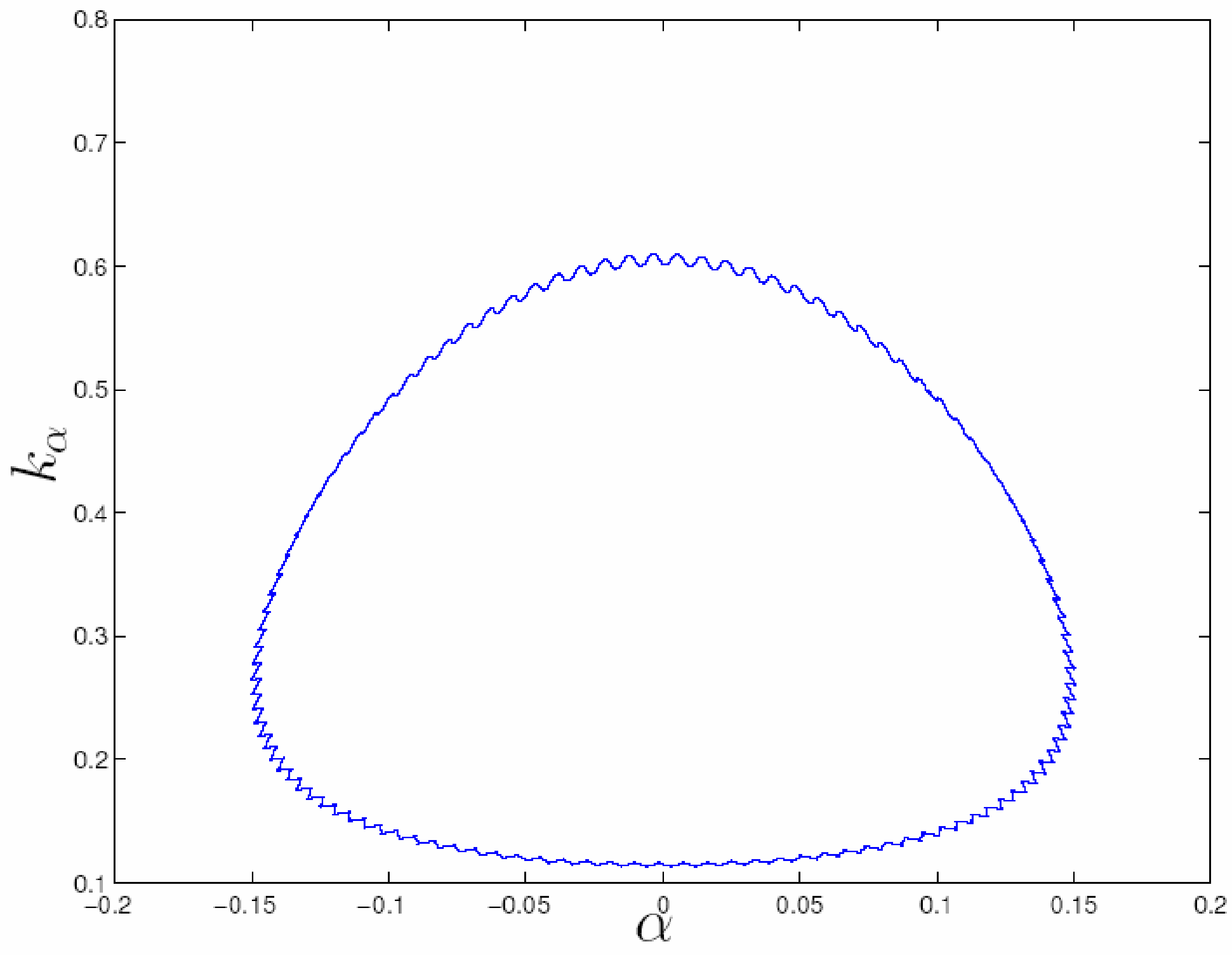
- For a stable mode require

$$k_\alpha^2\Omega_{*i}^2 + 4\lambda \geq 0$$

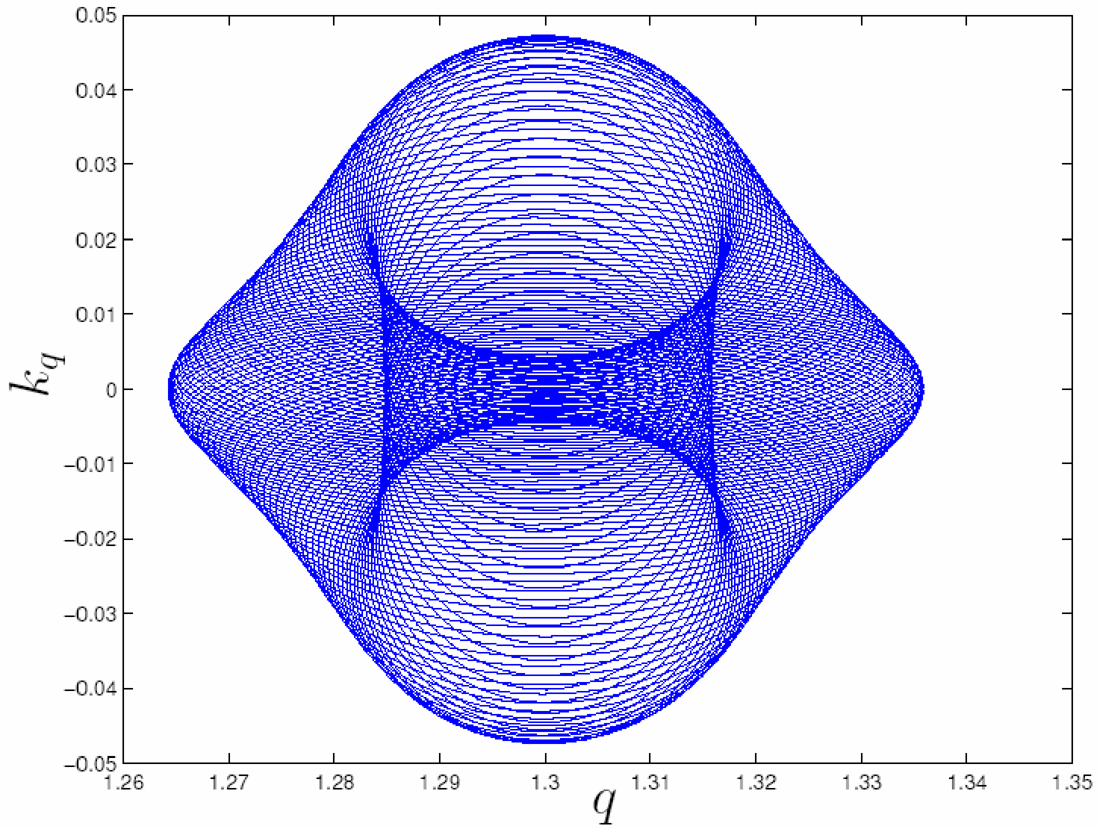
where neither k_α nor λ are constant

- Choose same $(\alpha_0, q_0, k_{\alpha 0}, k_{q0})$ with Ω_{*i} such that mode is marginally stable

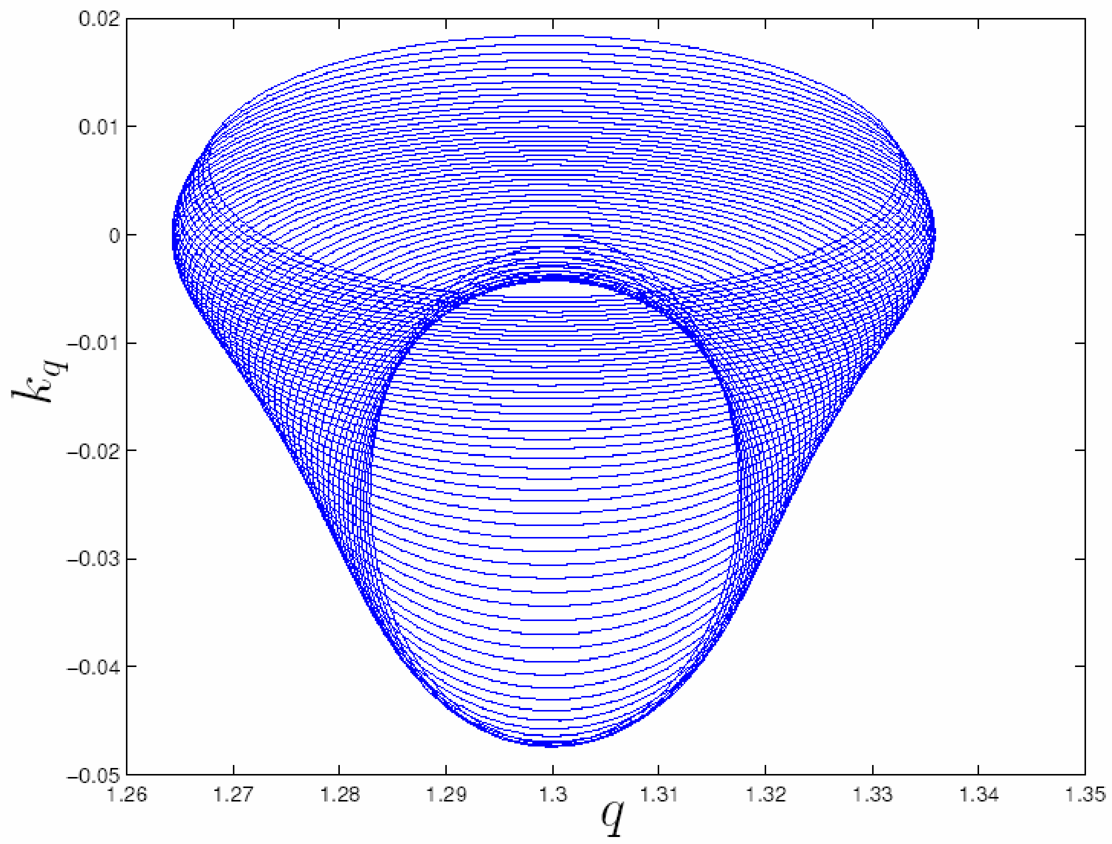
The projection of the ray equations into $k_\alpha - \alpha$ space shows closed orbits - quantizable action



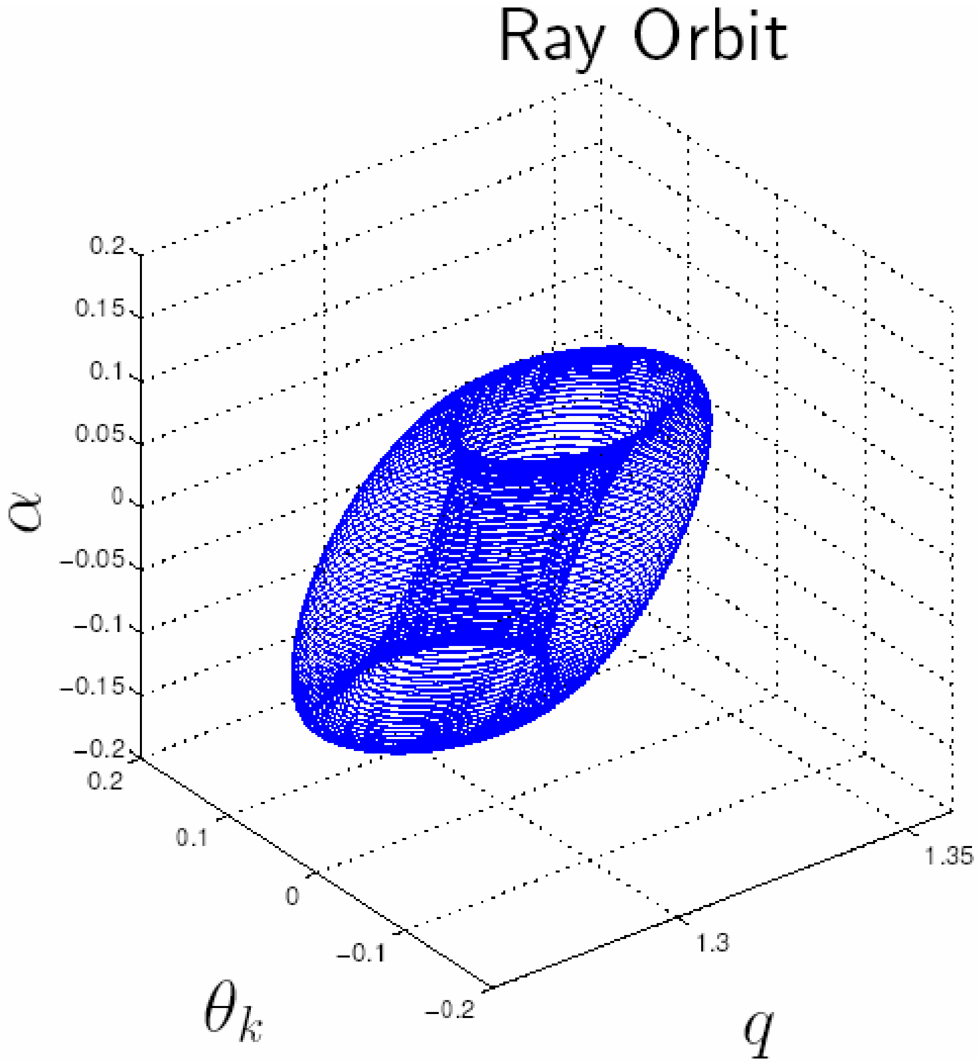
The projection of the ray equations into k_q–q space shows multiple "timescales"



Time scale separations typically allow for approximate integrability of the system



Inclusion of FLR eliminates the topological spheroids --- integrability?



Summary

- Inclusion of FLR effects into ideal MHD ballooning modes discretizes the spectrum.
- The inclusion of FLR physics on ballooning stability is complicated by the non-constancy of $\omega_{*i} \sim k_\alpha$ along the ray equations ("n" is not a good quantum number.)
- FLR stabilization is given by the criterion

$$k_\alpha^2 |_{\max} \Omega_{*i}^2 + 4\lambda_o > 0$$

  - $k_\alpha|_{\max}$ corresponds to peak value on periodic ray orbit
  - λ_o is the corresponding ideal MHD eigenvalue ($\lambda = \omega_{MHD}{}^2$).