Magnetics measurements in NCSX: SVD/PCA methods-I

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Introduction

A variety of magnetic diagnostics (MD's) will be installed in NCSX and used for

- Shape Control (during shots)
- Equilibrium Reconstruction (between shots)

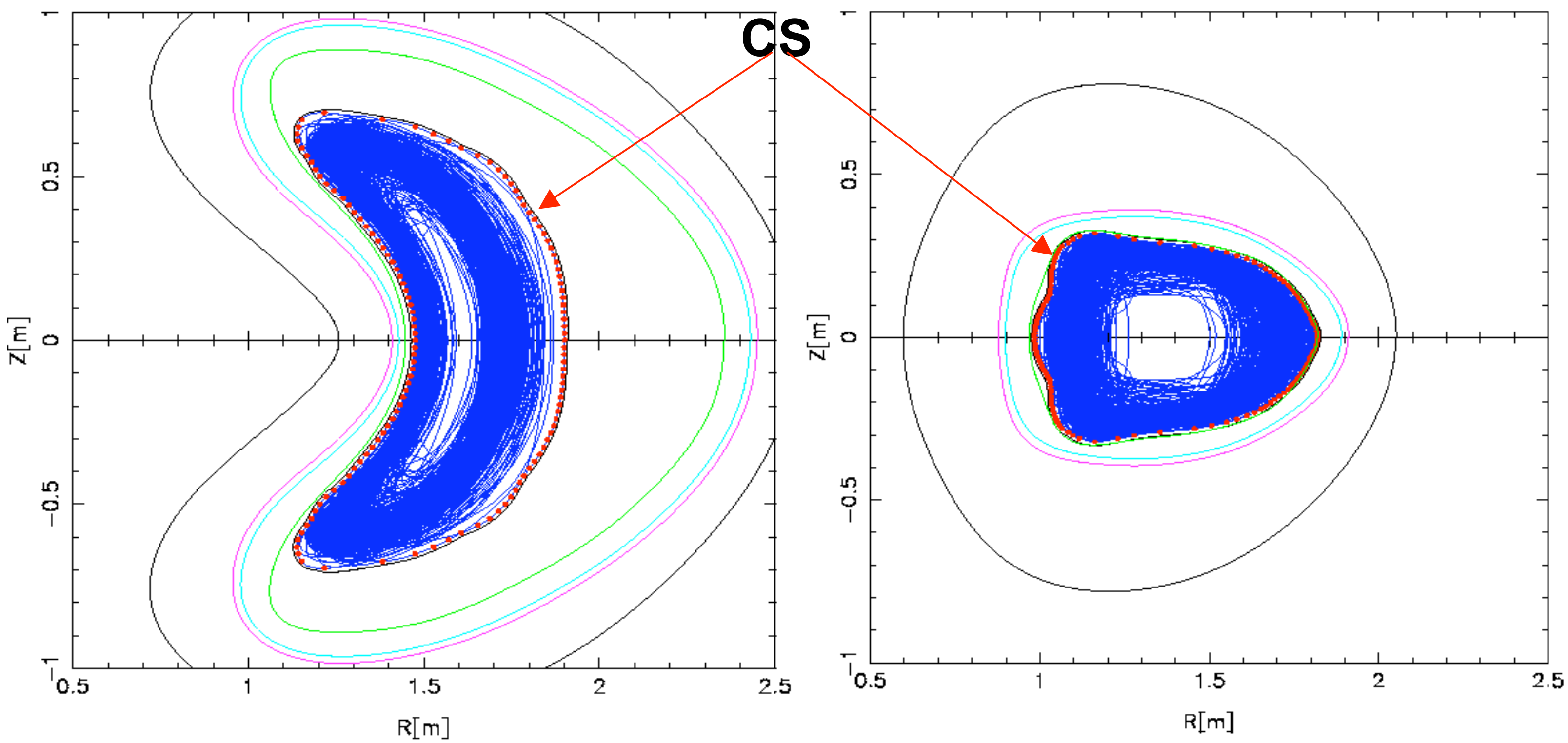
1) Define a method for selecting a "good" set of MD's (optimizes the invertible information)
2) How much info. is available directly from external MD signals? – more than tokamaks?

This presentation is a partial coverage of topic 2.

Target Function on Control Surface (CS)

- A database of VMEC equilibria (many hundreds) is being generated with a wide range of shapes and profiles.
- B-fields from each of the equilibria are calculated on a single "Control Surface" (CS) that lies 1cm outside the envelope of all equilibria. These are distributions, $b_j(q,f)$, j = 1…N_{eq}
- Magnetic signals $d_j(\mathbf{x}_d)$ are also calculated for each candidate diagnostic using V3RFUN and V3POST.
- The $b_j(q,f)$ are targets for the $d_j(\mathbf{x}_d)$. If the targets are reproduced with adequate precision the MD's should provide sufficient information for control and equilibrium reconstruction!

The Equilibria in the Database have a wide variety of shapes

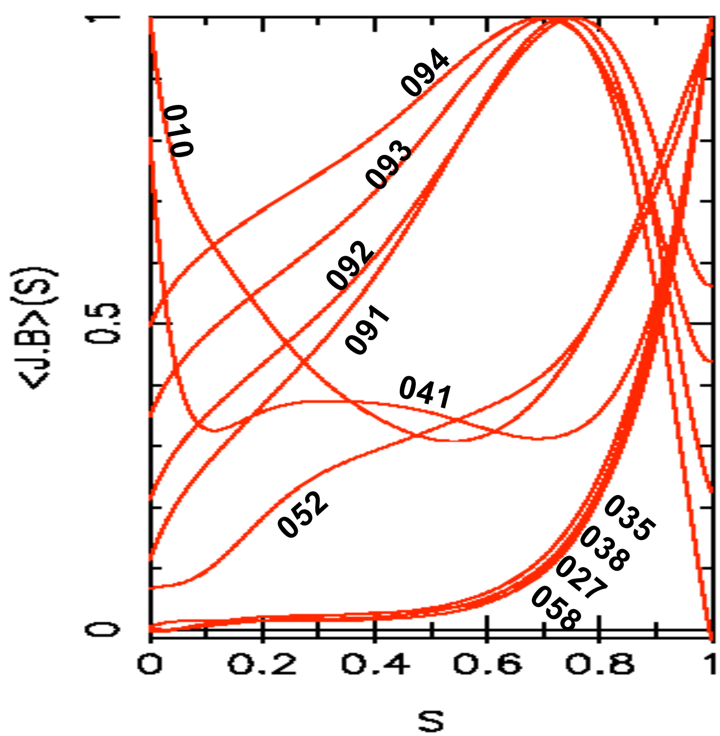


A surface is defined (the “Control Surface”, CS) which encloses all plasmas in the database. It lies 1cm outside of the envelope of all equilibria. The $B_\perp$ from each equilibrium is calculated by V3RFUN/V3POST and stored for analysis.

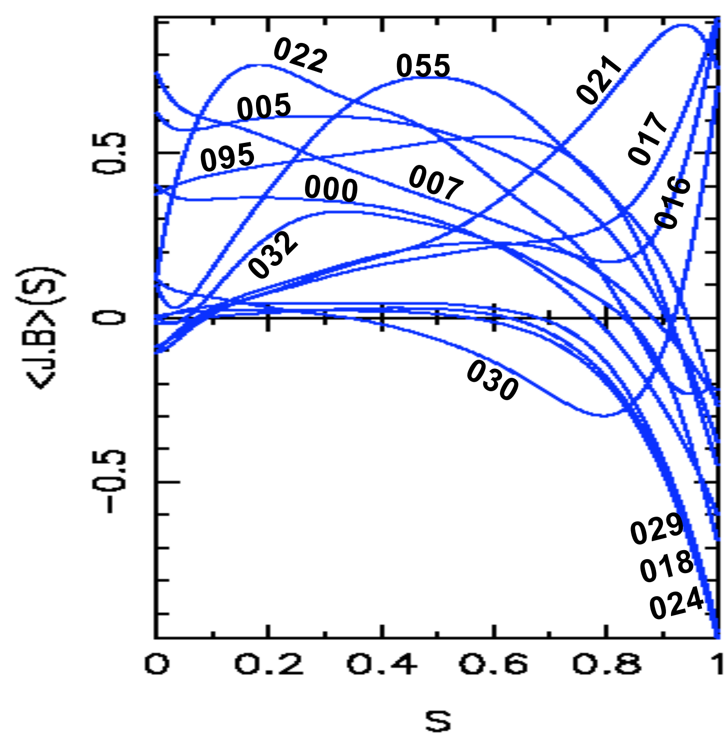
Current Profiles in Database

Profiles with <J.B>(s) > 0 for all s

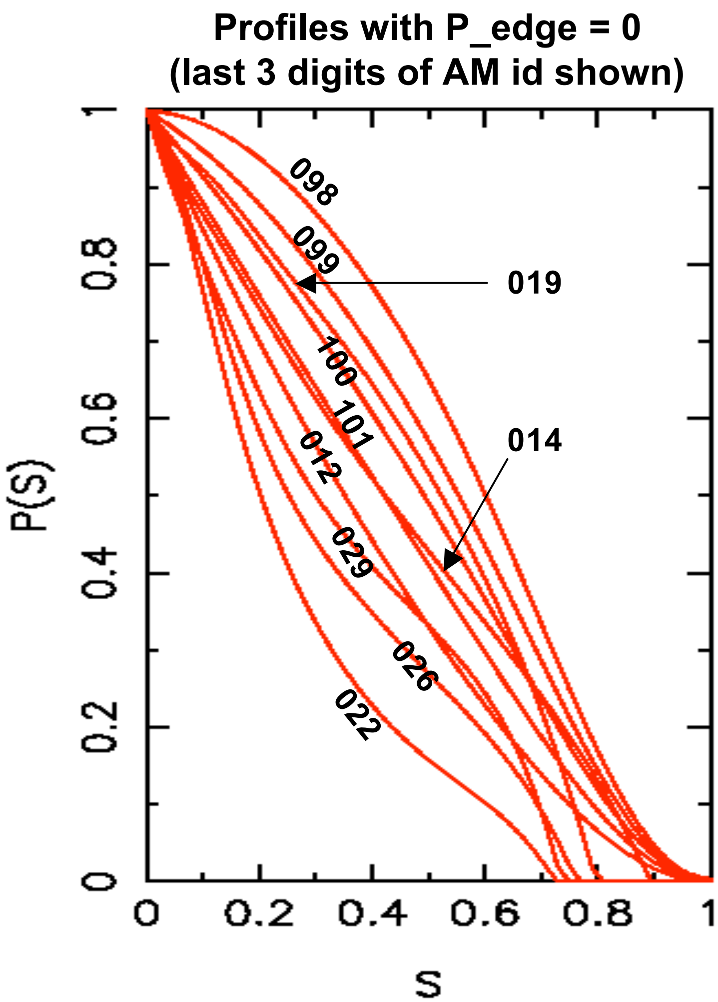
(last 3 digits of AC id shown)

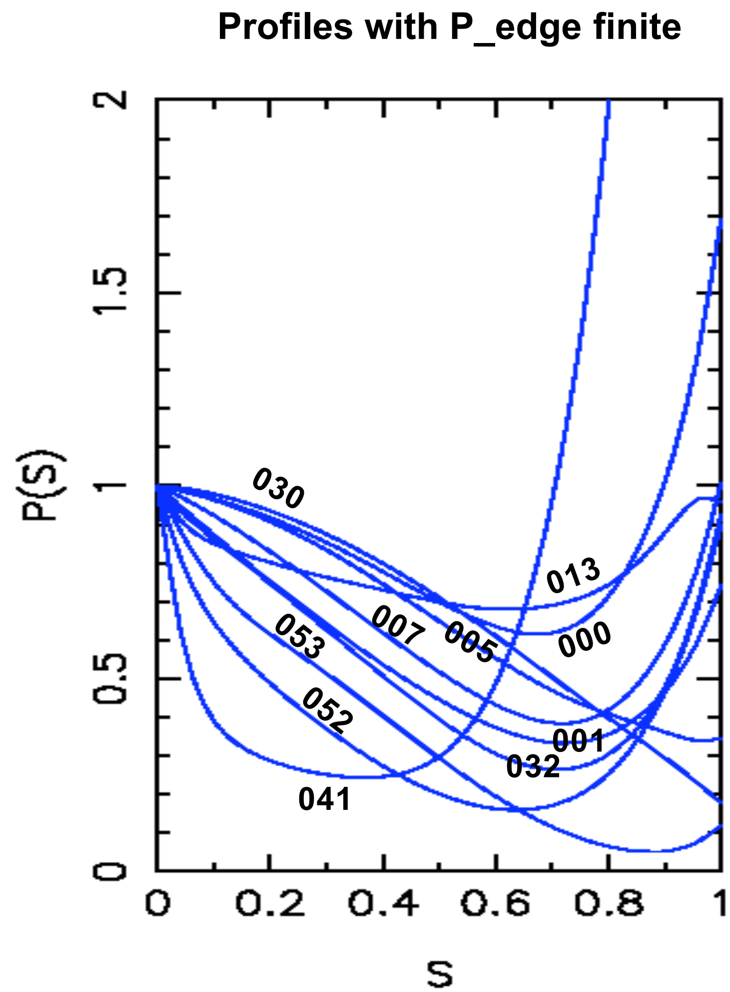


Profiles with <J.B>(s)
changing sign at some s

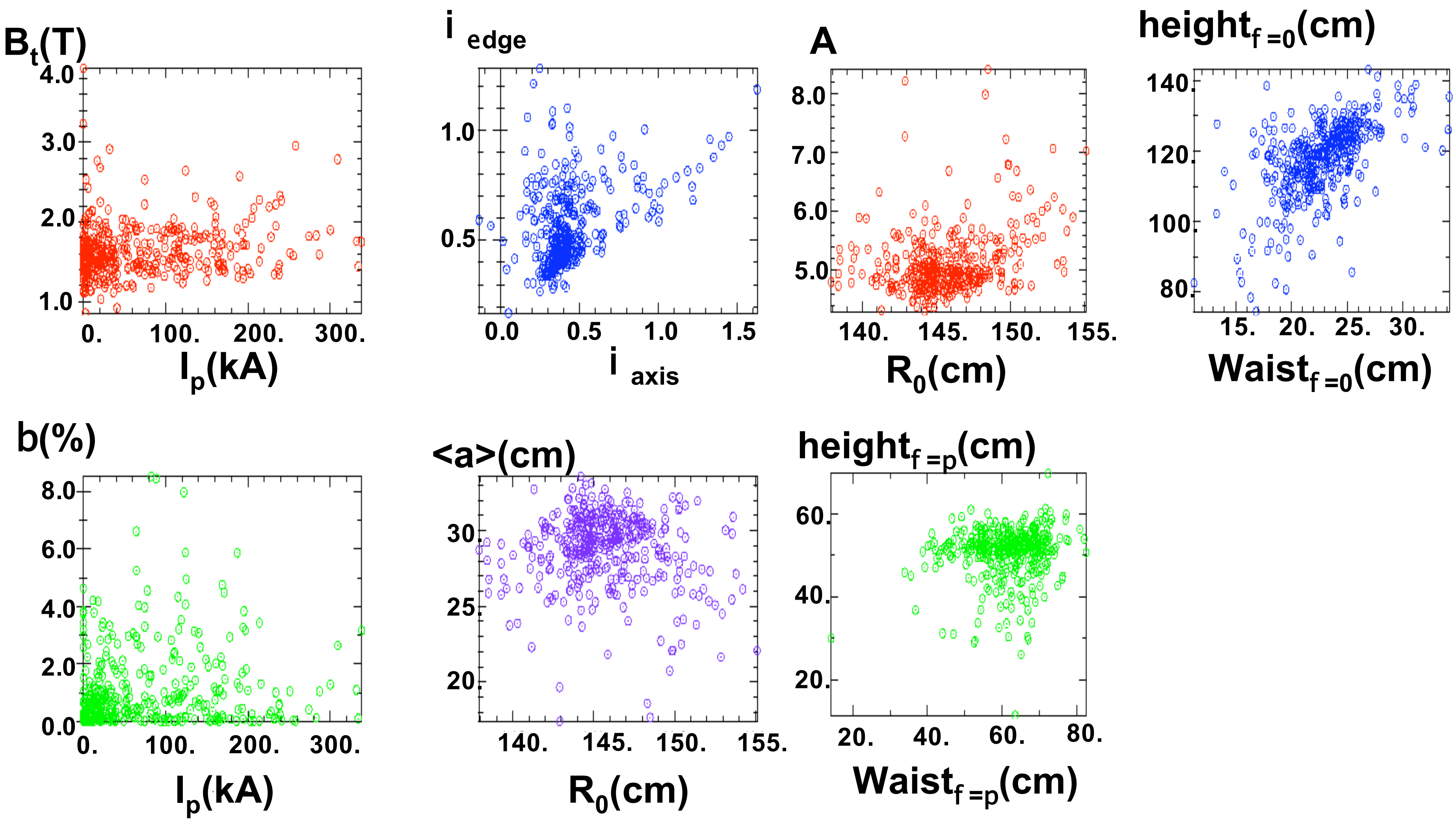


Pressure Profiles in Database





Equilibrium Database Parameters



Data Preparation and Expansion in EOF's

- For each equilibrium, labelled by index j, calculate $B_{\perp}$ on a uniform mesh of M points on the CS. (This is $b_j(q,f)$)
- Store the signal as an M-element column vector $\mathbf{x}_j$.
- Data from N_{eq} equilibria naturally forms an $M \times N_{eq}$ matrix, $\mathbf{X}$.
- Each column of $\mathbf{X}$ ($B_{\perp}$ signal on the CS) has an <u>exact</u> expansion as a linear combination of $\min(M,N_{eq})$ orthogonal patterns (called Empirical Orthogonal Eigenfunctions (EOFs)
- The EOFs are eigenfunctions of the correlation matrix $\mathbf{C} = \mathbf{X}\,\mathbf{X}^{\mathbf{T}}$.
- The calculation of EOFs is most conveniently done by Singular Value Decomposition of $\mathbf{X}$

Singular Value Decomposition (SVD), Principal Components, and EOF's

SVD: $\mathbf{X}_{MxN_{eq}} = \mathbf{U}_{MxM}\,\mathbf{W}_{MxN_{eq}}\,\mathbf{V}^T_{N_{eq}xN_{eq}}$

equivalent to

$$\mathbf{x}_j = \sum_{k=1}^{M} (V_{jk} w_k)\,\mathbf{u}_k$$

$$\equiv \sum_{k=1}^{M} Z_k(j)\,\mathbf{u}_k$$

j[th] column of X

k[th] column of U: Empirical Orthogonal Functions (EOFs).

Score for j[th] observation on k[th] Principal Component (PC)

Note: $\mathbf{x}_j^{(\ell)} = \sum_{k=1}^{\ell} Z_k(j)\,\mathbf{u}_k$, minimizes $e = ||\mathbf{x}_j - \mathbf{x}_j^{(\ell)}||$

Interpretation of Principal Components

- Since $$\mathbf{x}_j = \sum_{k=1}^{M} Z_k(j)\, \mathbf{u}_k,$$

orthonormality of the EOFs =>

$$\mathbf{x}_j \cdot \mathbf{u}_k = Z_k(j)$$

LHS is essentially an overlap integral.

For X = matrix of $B_\perp$ signals, Principal Component "score"

$Z_k(j) = \iint dq\,df\ B_\perp^{(j)}(q,f)\ u^{(k)}(q,f)$ measures importance of k^{th} EOF

in determining the shape of j^{th} equilibrium signal on the CS.

Interpretation (Cont)

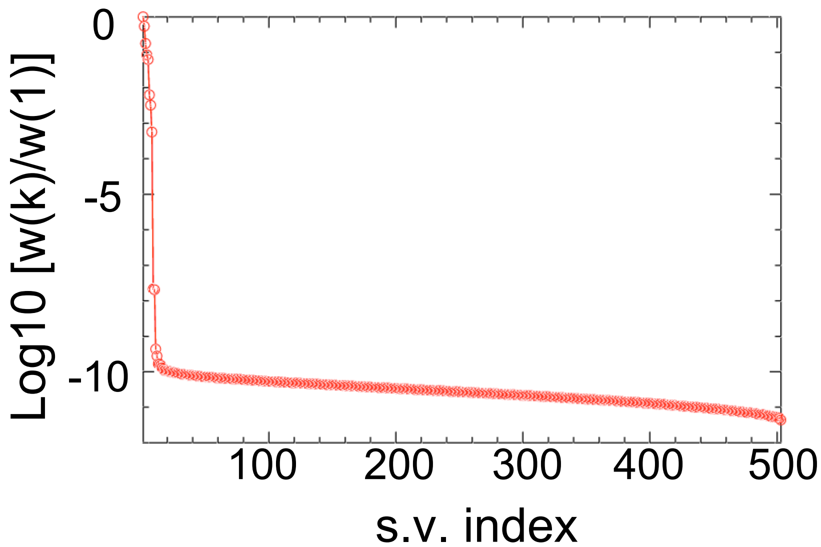
- **SVD on $X = U\,W\,V^T$ can be interpreted as a variable transformation $X := Z = U^T X$ $(= WV^T)$**

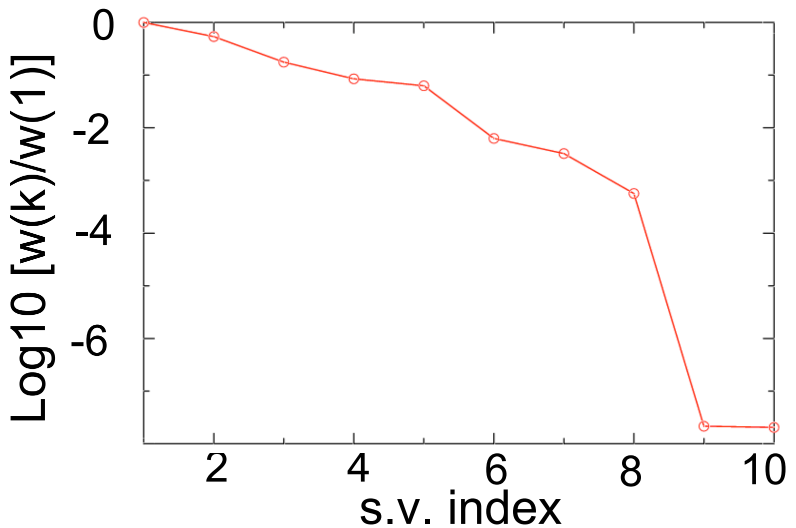
$$Z_{1j} = U_{11} X_{1j} + U_{21} X_{2j} + U_{31} X_{3j} + \ldots \;(= W_1 V_{j1})$$
$$Z_{2j} = U_{12} X_{1j} + U_{22} X_{2j} + U_{32} X_{3j} + \ldots \;(= W_2 V_{j2})$$
$$Z_{3j} = U_{13} X_{1j} + U_{23} X_{2j} + U_{33} X_{3j} + \ldots \;(= W_3 V_{j3})$$

… … … … … … etc … … … … … … … …

- **The U_{ij} are weights which measure the contribution of each of the original variables to the variance of the data in the transformed coordinates → means for selection/rejection of candidate magnetic diagnostics.**

Singular Value Analysis of Vacuum Signal ($B_\perp^{Total}$ - $B_\perp^{Plasma}$) on Control Surface

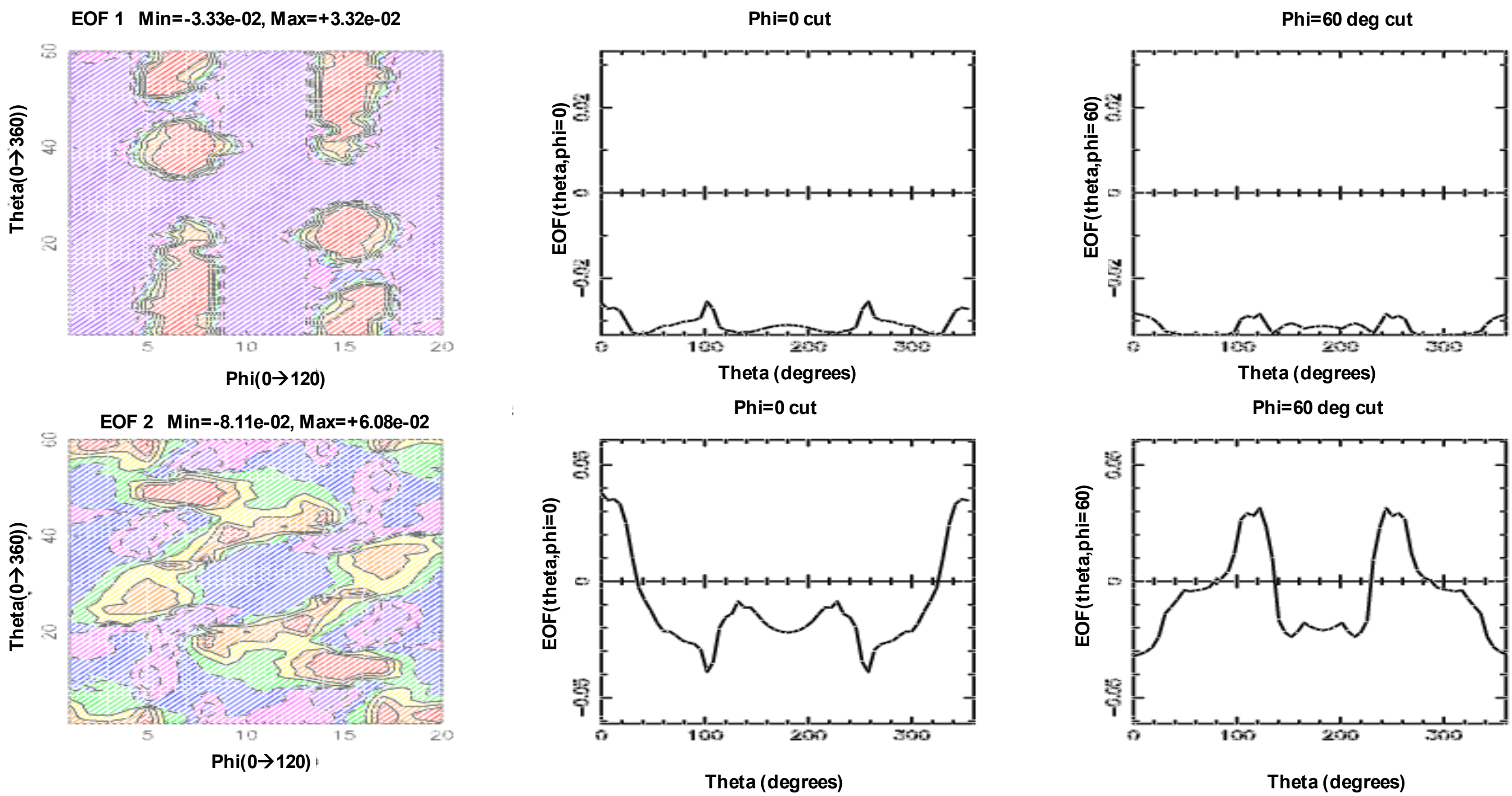




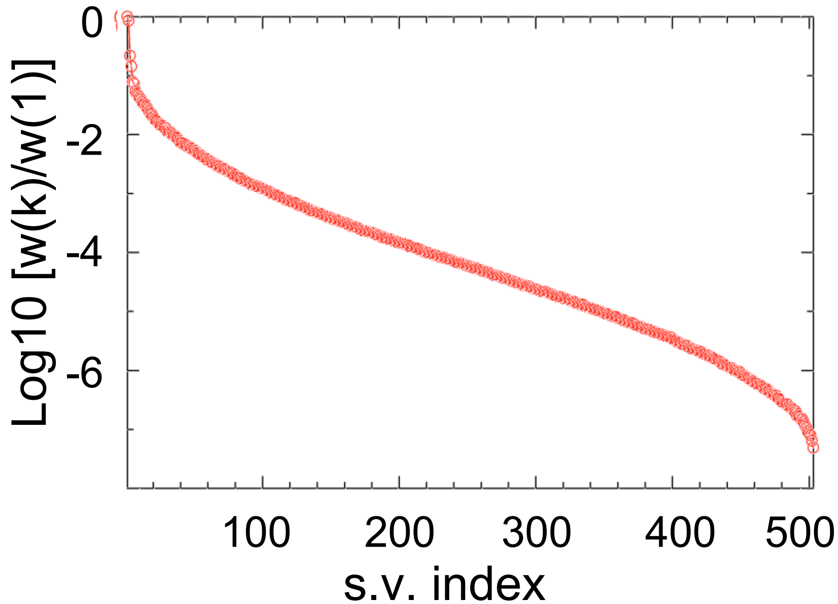
504 equilibria in database

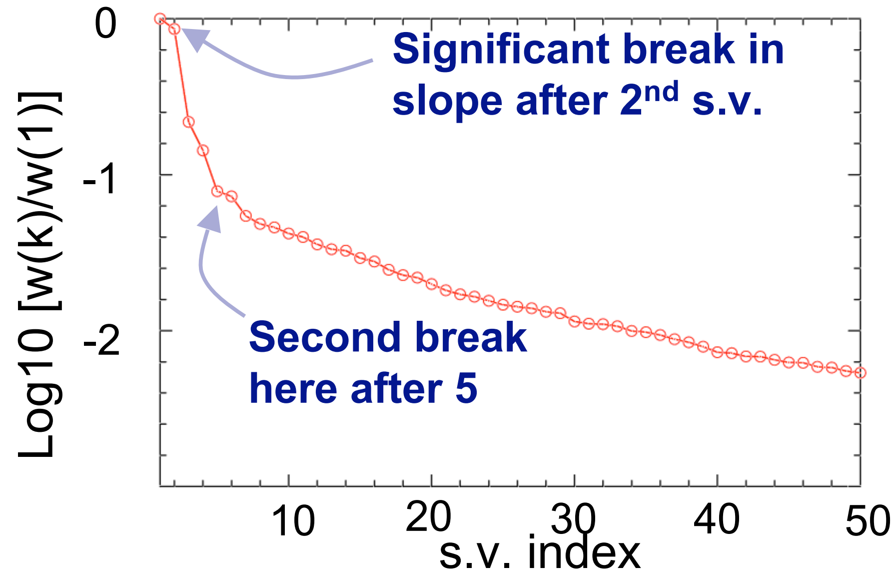
- **Find 8 significant singular values, corresponding to 8 independent, orthogonal, patterns of $B_\perp$ on the CS.**
- **The 8 patterns are to be expected because we have 8 equilibrium coil current groups (M1-3, TF, PF3-6).**

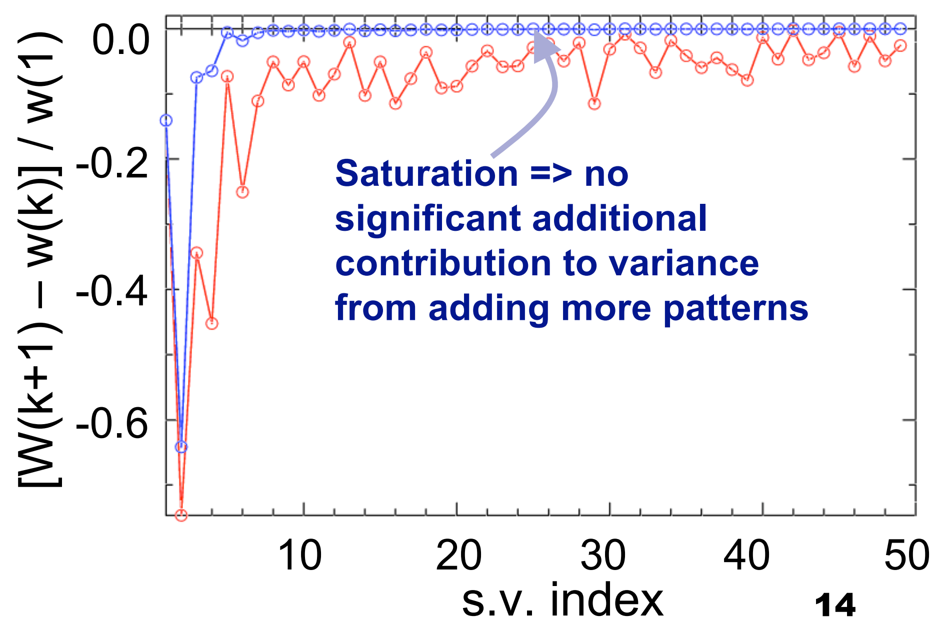
Correlation EOF's for Vac. Signal on CS (#s 1, 2)



SVD Analysis of $B_{\perp}^{Plasma}$







First 2 dominant patterns of $B_{\perp}^{plasma}$ contribute 94.5% of total variance. Next 3 contribute an additional 4.0%.

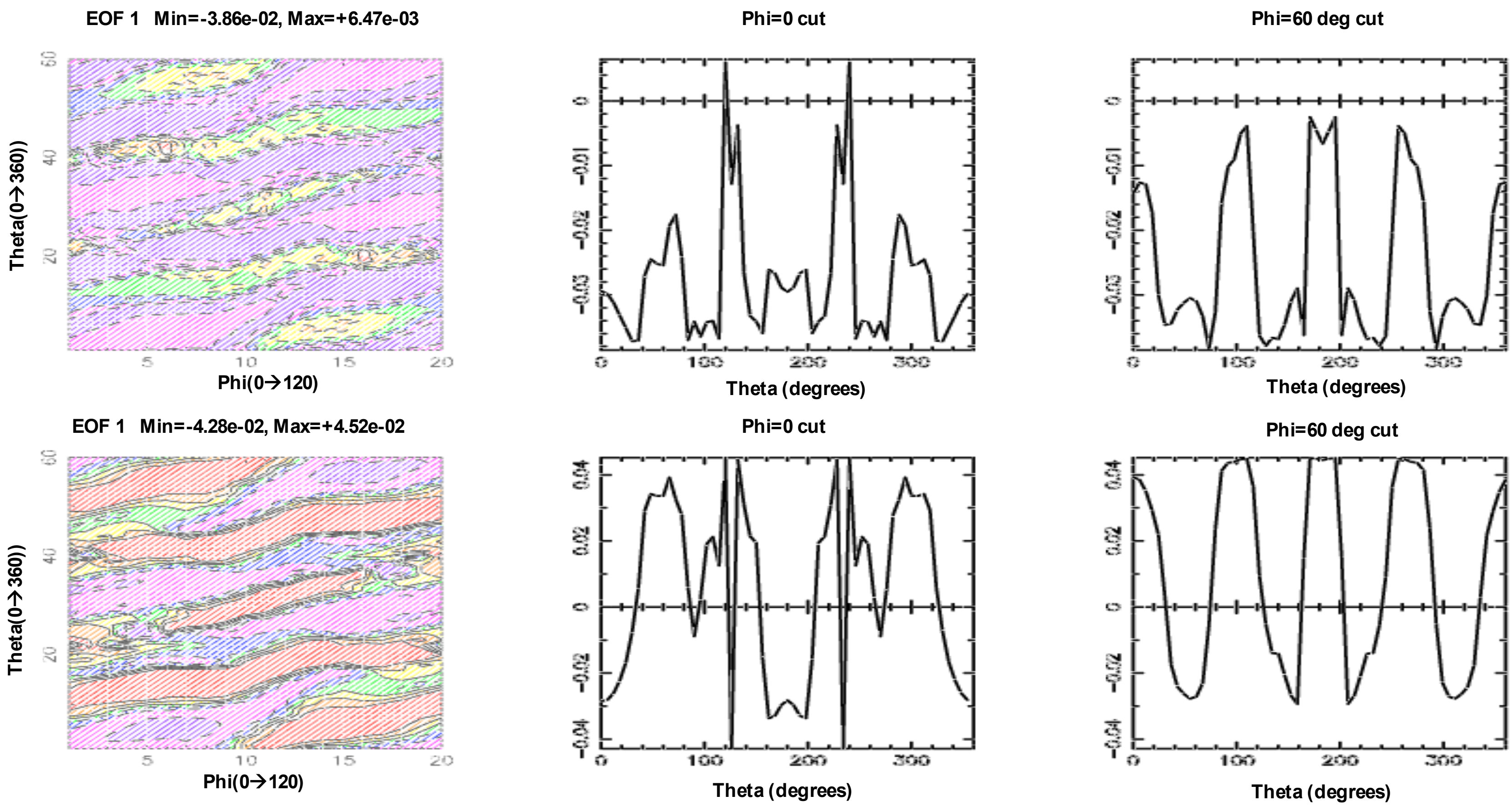
Speculate 2, and possibly up to 5, combined moments of p(s) and <J.B>(s) may be measurable.

How many independent pieces of information (PC's) are available from the Data Matrix?

Several procedures (mainly ad hoc)

- **Broken Stick Rule:** Retain principal components for which $L(k) = (w_k/w_1)^2 > 1/N \sum_{j=k}^{N} j^{-1}$ (expected length of k[th] longest segment of a stick of unit length broken at random into N segments)
- **Average Based:** Retain PC's for which $w_k^2 > 0.7 < w_j^2 >$
- **Cumulative Variance Based:** Retain as many PC's as necessary to bring the cumulative variance of the retained PC's up to some desired value, say 95% of the total variance – i.e., find k such that $\sum_{j=1}^{k} w_j^2 / \sum_{j=1}^{N} w_j^2 > 0.95$

Correlation EOF's for Plasma Signal on CS (#s 1, 2)



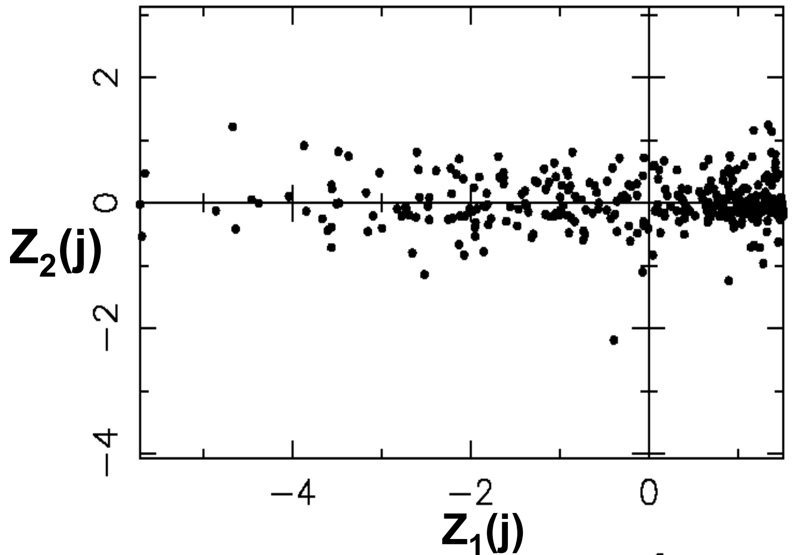
More structure (shorter wavelength) seen here than for vacuum signal – simply distance attenuation?

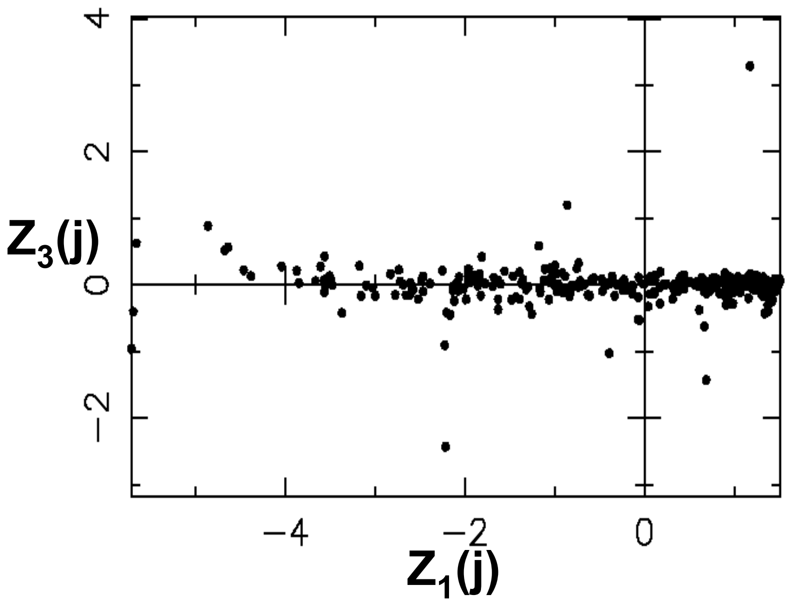
PC Scores (All Equilibria)

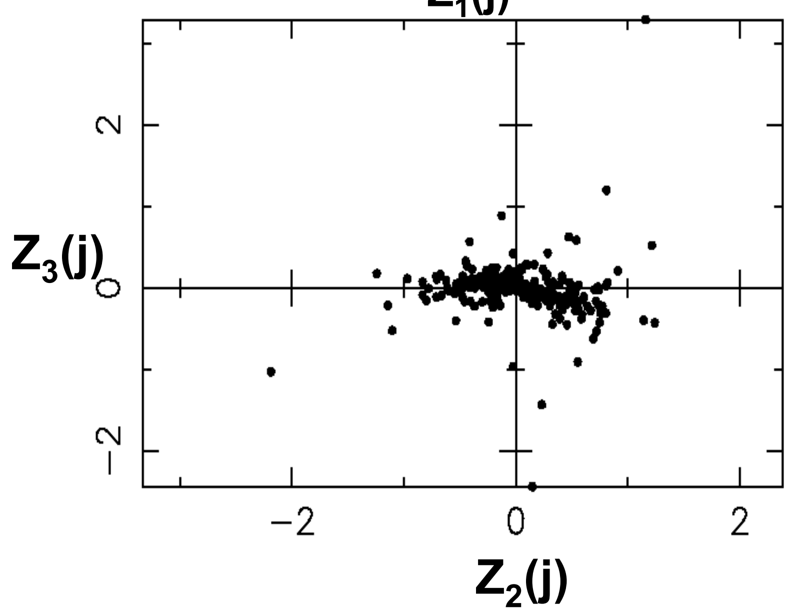
Each dot represents a particular equilibrium.

All equilibria in the 07/15/04 database are shown.

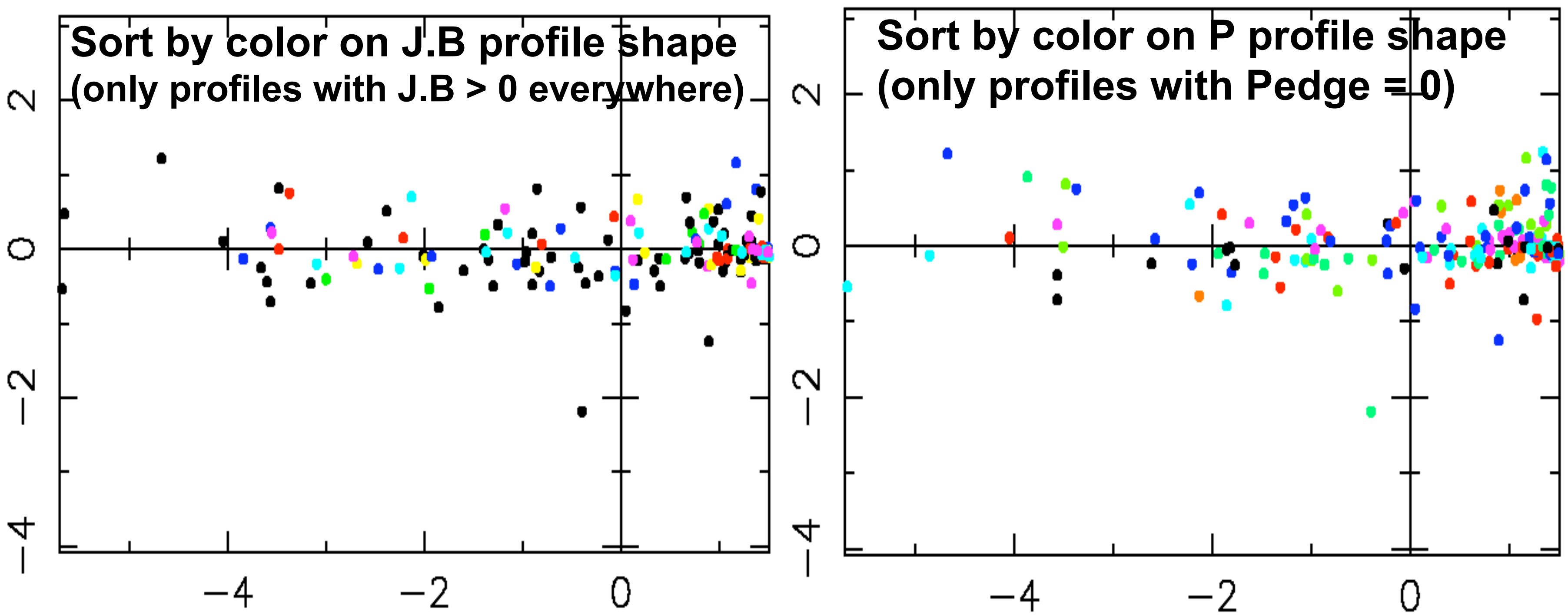
As an attempt to discover what properties of the equilibria are responsible for the dominant $B_\perp$ signal patterns, try color coding.





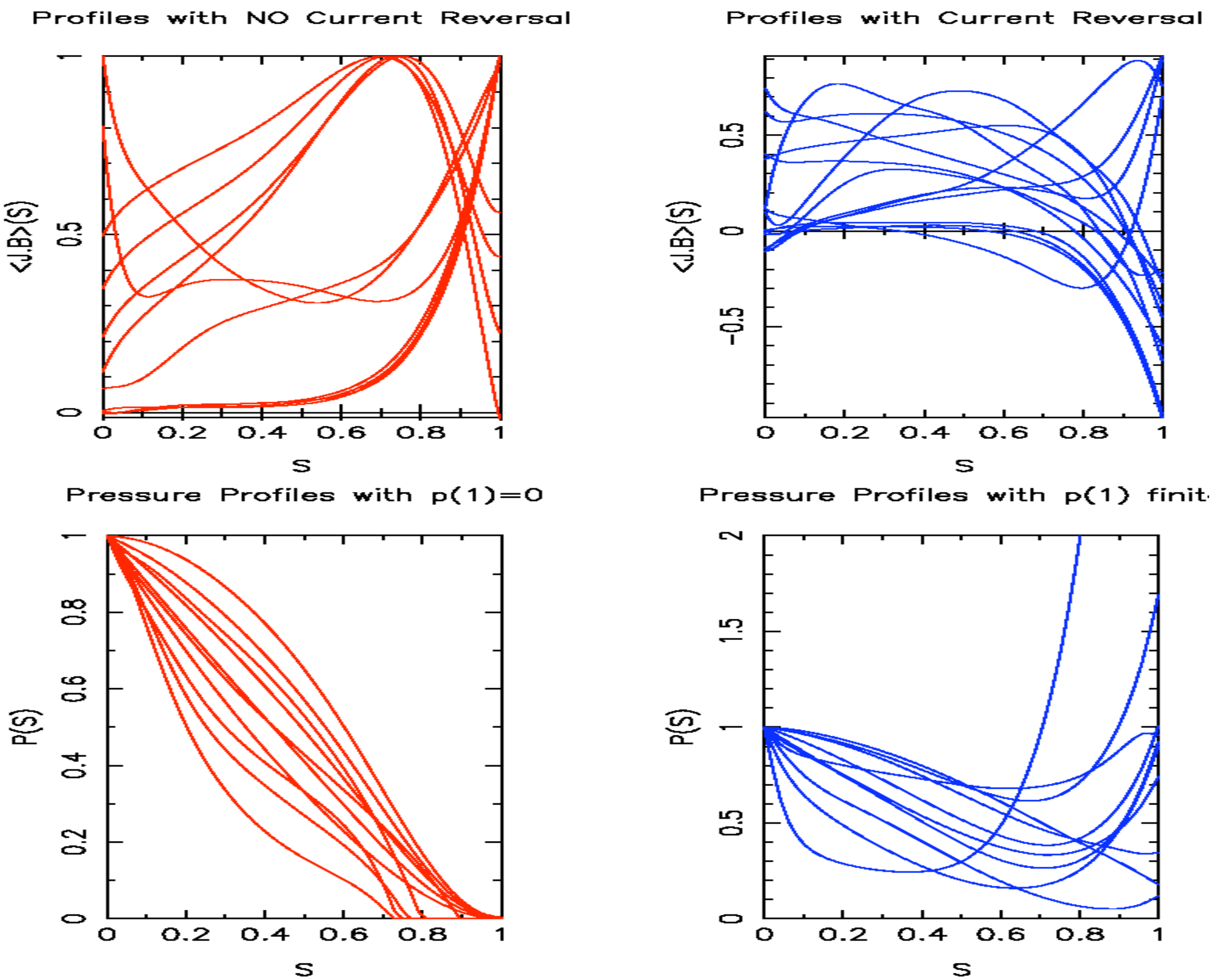


PC2 vs PC1 with Color Coding for Various Plasma Properties

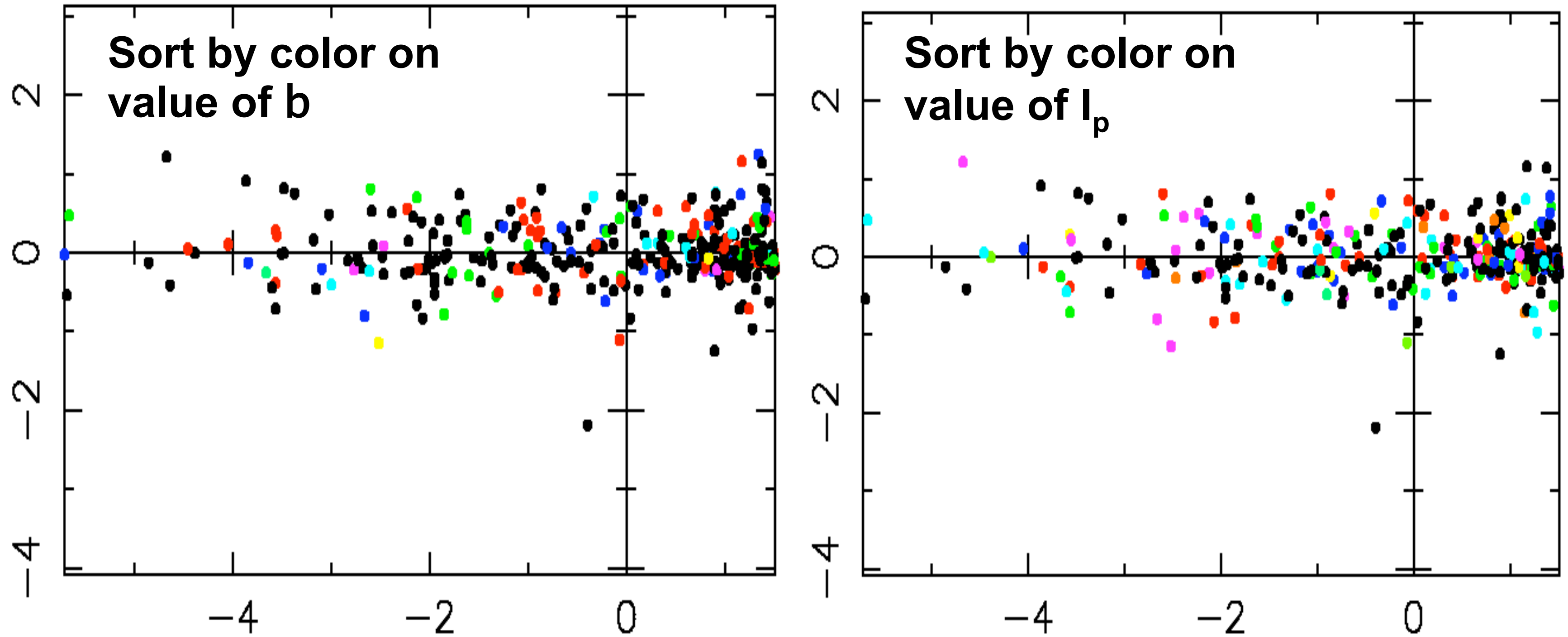


Would like to see a separation of colors along one or other of the axes, thereby associating a particular pattern of $B_\perp$ with a profile parameter variation. A sensible guess for the appropriate profile parameters for color separation are $\ell_i/2$ and b_{pol}

Current and Pressure Profiles in Database



PC2 vs PC1 with Color Coding for Various Plasma Properties

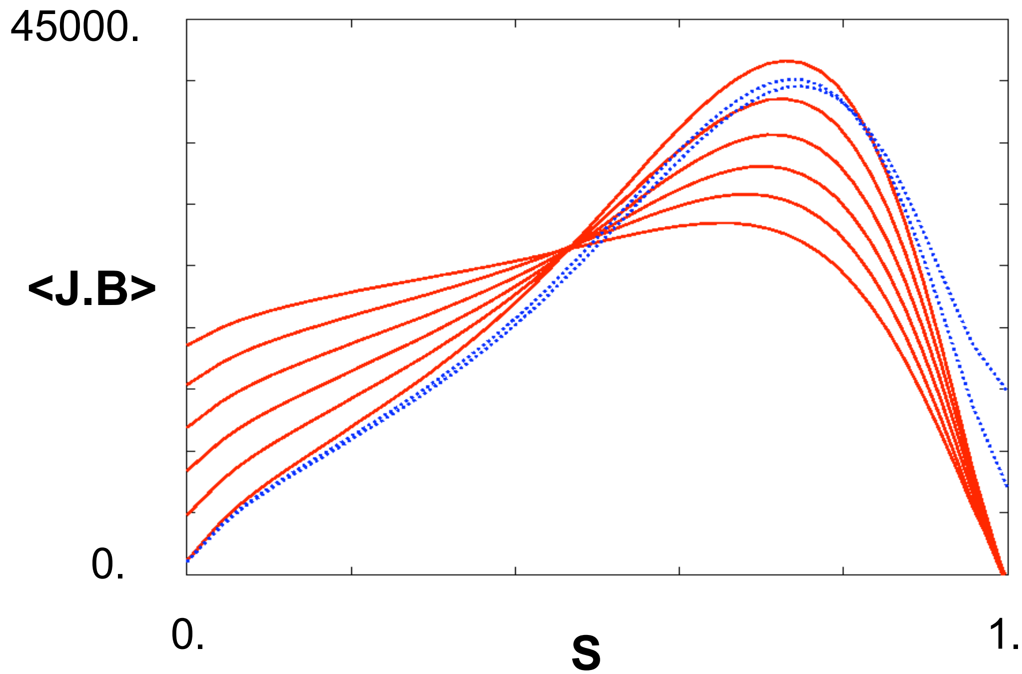


b and I_p not individually responsible for the two dominant $B_\perp$ patterns.

Why should we think any of this is possible?

- Because of results obtained from a similar analysis using equilibria from the plasma flexibiliity studies made in preparation for the CDR.   There, we had a much more limited set of equilibria, but with the advantage of systematic variation of profiles and plasma parameters.

Can we detect current profile variation from external magnetic measurements?

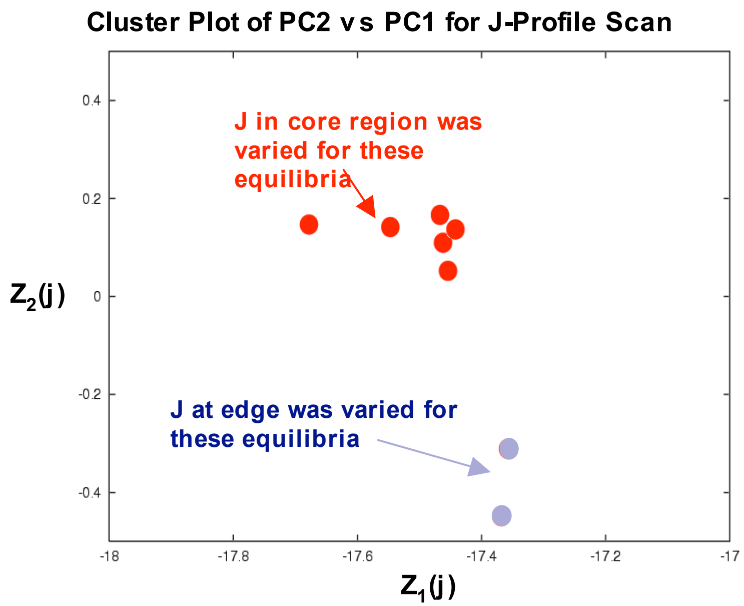


As a test case, consider magnetic measurements for 2 groups of equilibria where the current profile is varied at fixed I_p and b.

1st group: 6 equilibria where <J.B> is varied in core region (red)

2nd group: 2 equilibria where current is added to the edge (blue)

PCA provides a method for distinguishing equilibria



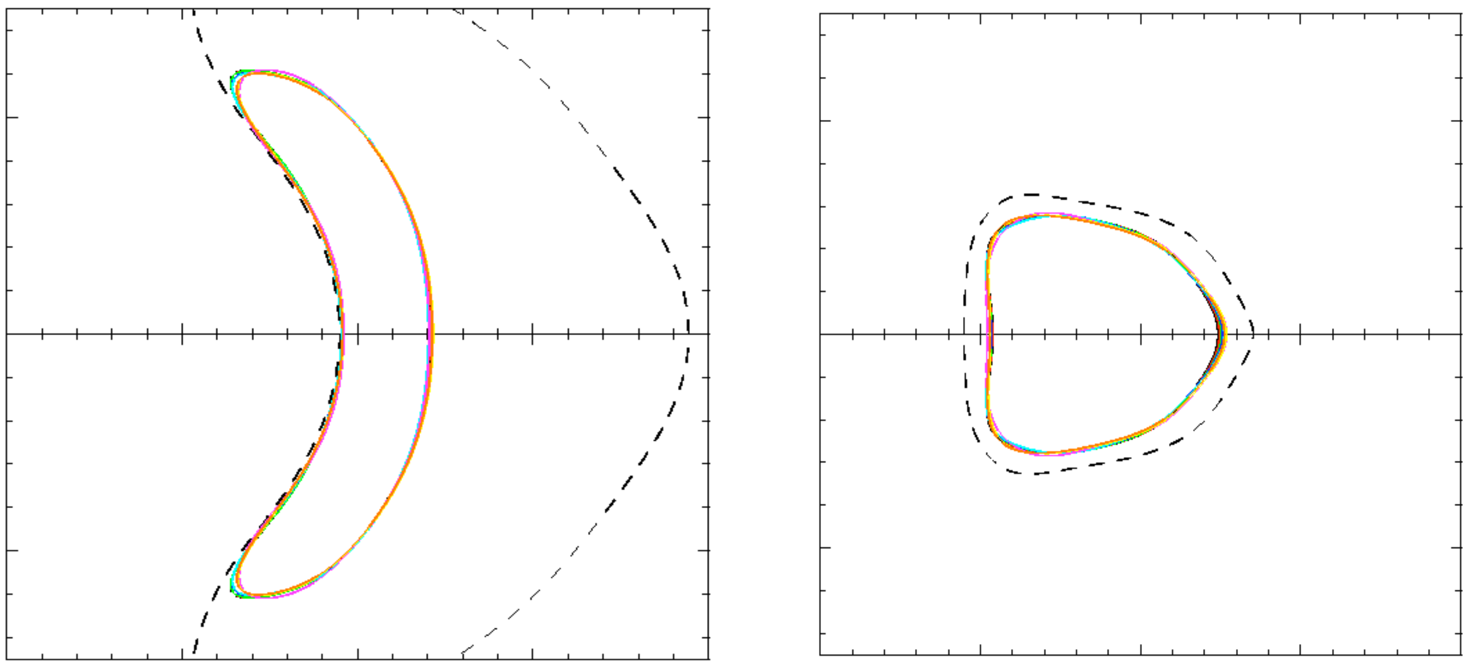
- The $B_\perp$-matrix on the CS is analysed by Singular Value Decomposition.

$$X = U\,W\,V^T$$

- According to this decomposition, the columns of U (denote by $\mathbf{u}_k$) provide a basis for the expansion of any of the field patterns (columns of X) on the CS.
- The $\mathbf{u}_k$ are called Empirical Orthogonal Functions (EOF's) and the coefficients of the $\{\mathbf{u}_k\}$ are called Principal Components.
- A linear combination of the first few EOF's can describe much of the variation in the data.

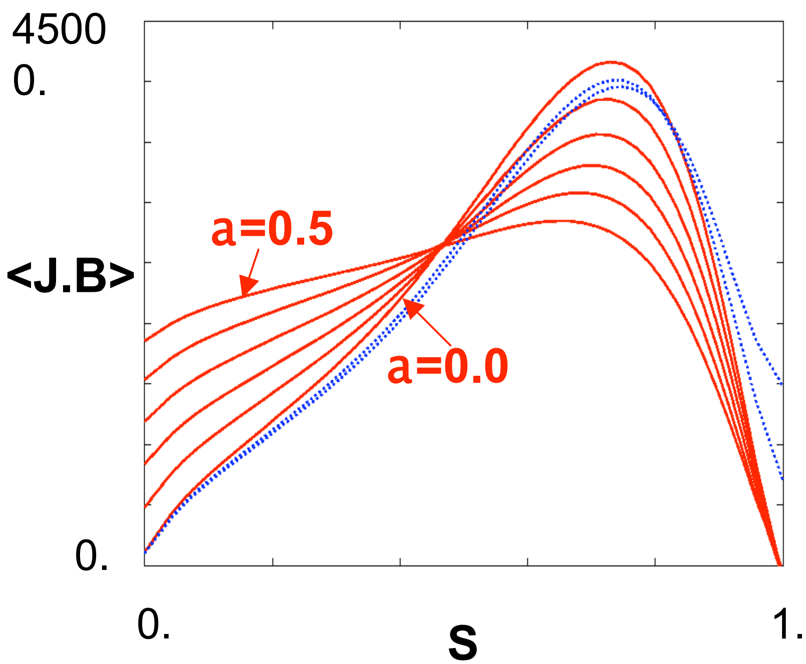
A 2D scatterplot of the first two PC's of the $B_\perp$-matrix data corresponding to the 8 equilibria in the J-profile scan distinguishes equilibria where the current profile was varied in the core (red cluster) and equilibria where edge current was added (blue cluster).

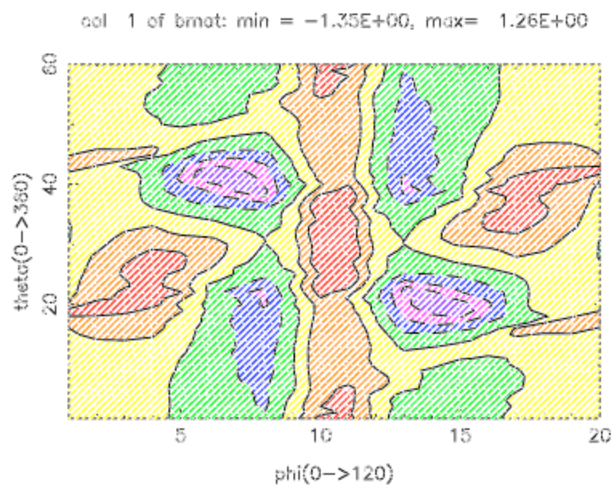
Note: The plasma boundary shape variation is very small between these 8 equilibria



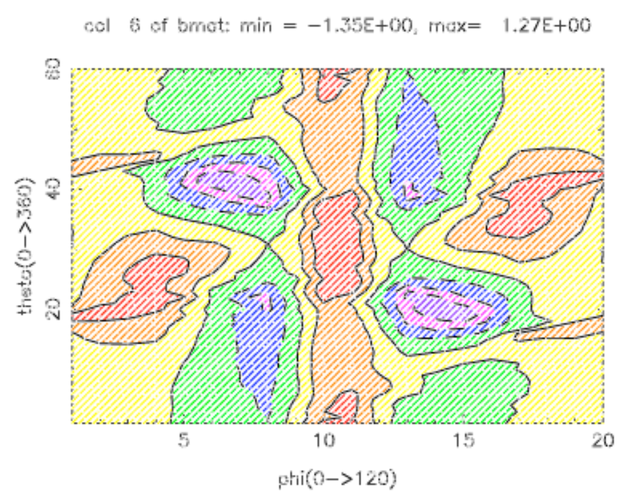
• Therefore if analysis of the B_q signals on the CS can distinguish between these equilibria, it is mainly due to the profile variation, not the consequent shape variation.

$B_\perp$ signals on Control Surface for equilibria with different <J.B>-profile shapes (fixed I_p and b)





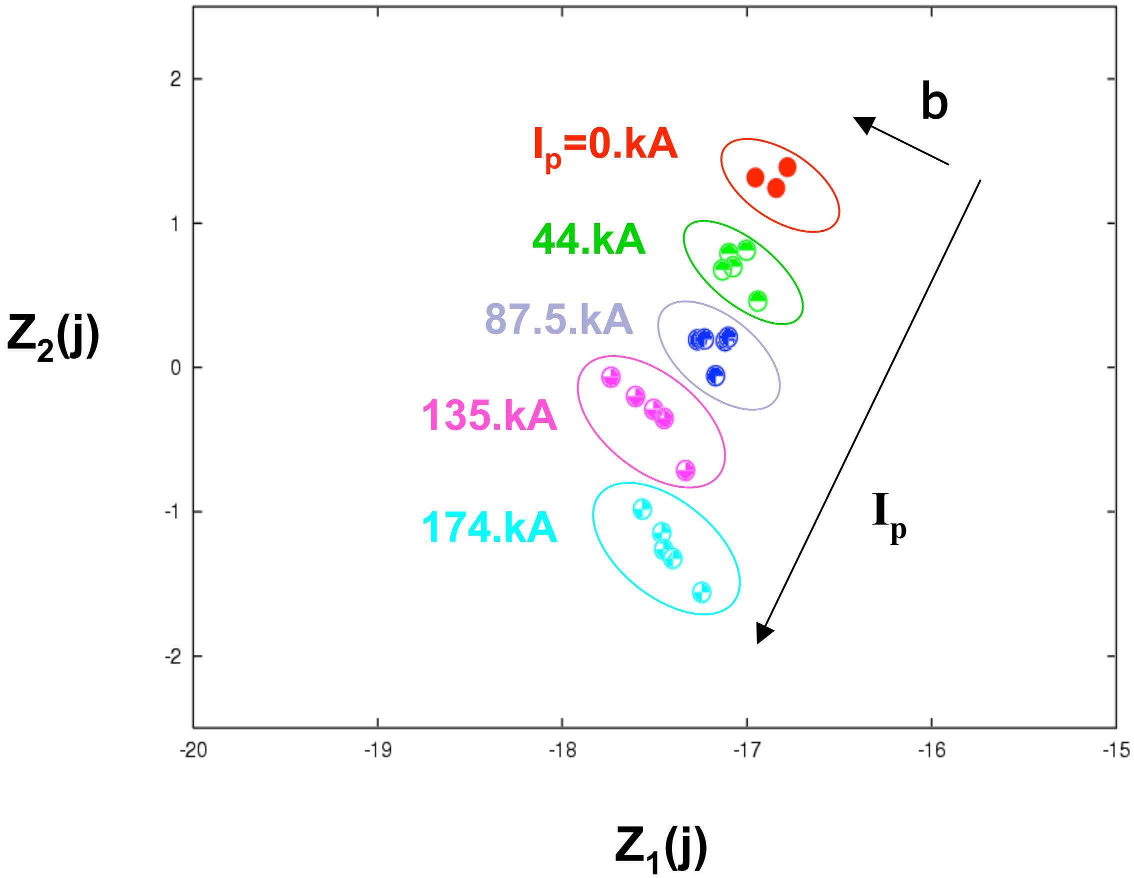
a=0.0



a=0.5

- The $B_\perp$ pattern change is subtle.
- Principal Component Analysis (PCA) can distinguish the equilibria.

Projection onto plane of the leading 2 principal components separates an I_p-b equilibrium sequence



23 equilibria with constant <J.B>(s) and p(s) profiles

Difference between $B_\perp$ signals on CS for a=0.0 *and* a=0.5