Ideal MHD Ballooning Stability in Three-Dimensional Equilibria

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Theses

- An analytic tool has been developed to examine the effect of 3-D shaping and profiles on the local properties of a 3-D configuration – ‘Local 3-D equilibria’

- The technique is illustrated for use in ideal MHD ballooning studies. Generic features of ballooning stability in 3-D configurations are addressed.
  - Ideal ballooning stability boundaries
  - Unique features of 3-D - field line dependence of local eigenvalues
  - Geometric interpretation
  - Implications for global mode stability

Outline

- Introduction and motivation
- Local 3-D MHD equilibria
- Ideal Ballooning Studies
  - Stability boundaries (generalized s-α curves)
  - 3-D features
- Summary

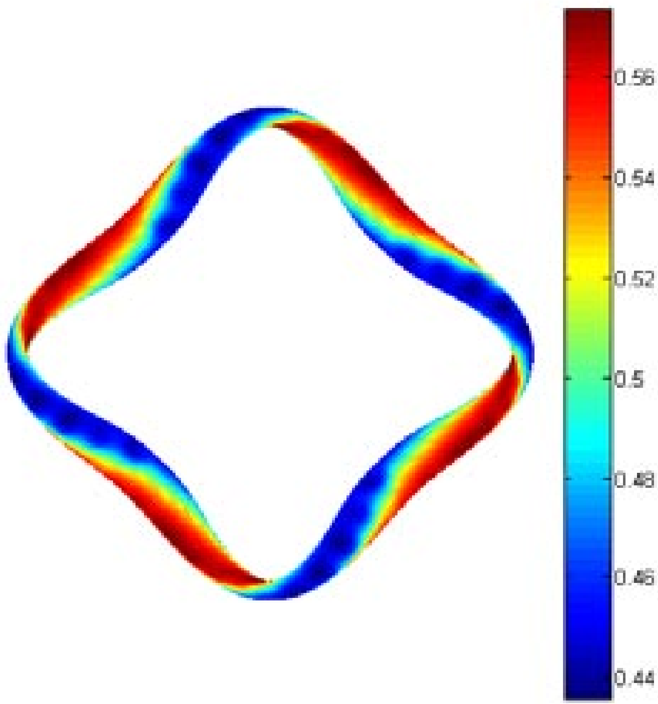
Motivation

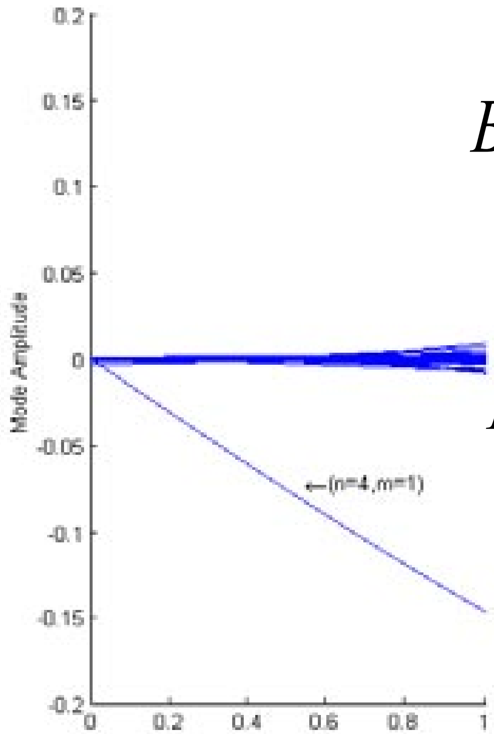
- Theoretical insights into confinement improvement are anticipated to lead to improved transport properties in stellarators.
- As confinement improves, β limits for stellarators become an issue.
- Since most stellarators operate with little current,
  - Current driven kinks/tearing modes should not be an issue.
  - NTM's can be avoided with the proper choice of rotational transform profile.
- The violation of the Mercier criterion does not seem to influence confinement in stellarators or tokamaks (CHS, LHD, TFTR, JET, etc.).
  - Growth rates slightly above marginality are feeble (S. Gupta, et al '02)
  - Easily stabilized by finite gyroradius effects.
  - Nonlinearly, structures at low-order rational that flatten-p (Ichiguichi et al, '02)
- Ballooning stability is a possible candidate for producing a β-limit.

The HSX Device

Major Radius	1.2 m
<r>	0.11 m
Volume	~.44 m^3
Field Periods	4
ι_{axis}	1.05
ι_{edge}	1.12
Coils/period	12
B_0 (max.)	1.25 T
Magnetic Field Flattop Length	0.2 s
Auxiliary Coils	48

The Helically Symmetric eXperiment





Quasihelical: Fully 3-D, BUT Symmetry in |B| :

$$B = B_0\left\lfloor 1 - \varepsilon_h \cos\left(N\phi - m\theta\right)\right\rfloor$$

In straight line $\theta = \iota\phi$ coordinates ,

$$B = B_0\left\lfloor 1 - \varepsilon_h \cos\left(N - m\iota\right)\phi\right\rfloor$$

In HSX: N=4, m=1, and $\iota \sim 1$

$\iota_{eff} = N - m\,\iota = 1/q_{eff} \sim 3$

Ballooning stability is an issue for stellarators

- Ideal ballooning stability is often used as an indication for β-limits in stellarator designs. To a large degree, most ideal ballooning stability studies have been undertaken with specific designs in mind.
- Some of the existing devices have not been optimized to minimize ballooning stability issues. As such, many of the theoretically predicted β-limits from ideal ballooning are rather small, $\beta < 2\%$.
- However, exceptions to the small theoretical ballooning β-limits exist (W7-X, NCSX), so ballooning does not seem to be a fundamental problem for all stellarators.
- It is an open question experimentally, whether ballooning modes (or any MHD instability) really provides a rigorous limit.
- It would be desirable to obtain a deeper physical understanding of the structure of ballooning stability in stellarators.

Generating 3-D equilibria for stability calculation is non-trivial

- The starting point for any stability analysis is a well-defined equilibrium.
- While the equilibrium properties of axisymmetric configurations is well understood. - $\Delta^{*}\psi = - R^{2}p' - FF'$ - there is no general prescription for constructing global solutions to the MHD equilibrium equations in 3-D
  - Numerical methods - time consuming and unwieldy for parameter scans
  - Singular currents at rational surface
  - In general, islands are present.

A method for constructing "local" solutions of 3-D MHD equilibria has been developed

- The difficult aspect of studying 3-D shaping effects is the generation of equilibria. There is no rigorous Grad-Shafranov theory for constructing *global* solutions to the MHD equilibrium equations for 3-D systems.
- "Local" 3-D equilibrium = specification of the magnetic coordinates mapping [or inverse mapping **x**(ψ,θ,ζ)] in the vicinity of the magnetic surface $\psi = \psi_o$ to sufficient order to describe **J**,**B**, p' consistent with

  $$\vec{J} \times \vec{B} = \nabla p, \nabla \times \vec{B} = \mu_o \vec{J}$$

  One can explicitly construct solutions on the magnetic surface [see PoP **7**, (2000), 3921].
- The "small denominator" problem that plagues global solutions to the MHD equilibria equations in 3-D can be avoided. For local stability criteria, we are concerned with the vicinity of a particular magnetic surface.

Local axisymmetric equilibria are useful for local mode studies in tokamaks

- Parameter scans in tokamak equilibria can be performed by generating sequences of Grad-Shafranov solutions.
- However, numerical solutions are time consuming - other methods have developed to generate a series of MHD equilibria, localized to a magnetic surface, without recomputing the entire Grad-Shafranov solution.
  - Connor, Hastie, Taylor, shifted circle equilibria ('77) - no shaping
  - Greene and Chance (NF '81) developed a generalized method for constructing sequences of a equilibria on a particular flux sruface as functions of two profile parameters (generalized s-α model). This technique is now routinely used in tokamak studies.
  - Miller et al (PoP '98) generalized this technique by allowing for variations in axisymmetric shaping parameters.
    - Used for studies on the effect of shaping on localized MHD instabilities and microinstabilities (Waltz and Miller, PoP '99).
- This work has been generalized to 3-D systems

Local 3-D equilibria are specified by two sets of quantities

- Local solution to MHD equilibria at $\psi = \psi_o$ specified by:
  - $\mathbf{X}(\theta,\zeta)$ and ι_o on $\psi = \psi_o$ (subject to constraints) where θ and ζ are any straight field line coordinates. Inverse mapping is convenient.
  - Two flux function profiles $$\frac{dp}{d\psi}, \frac{d\iota}{d\psi} = -\frac{1}{q^2}\frac{dq}{d\psi}$$

$\rightarrow$ Allows for easy manipulation of plasma profiles and 3-D shaping.

$\rightarrow$ Allows one to build intuition of the effect on local stability criterion; construction of stability boundaries as functions of profiles (s-α curves) and 3-D shaping parameters.

$\rightarrow$ Allows one to examine the sensitivity of a given configuration to changes in the equilibrium.

The choice of $\mathbf{X}(\theta,\zeta)$ determines geometric properties of local equilibria

- From $\mathbf{X}(\theta,\zeta)$ and ι_o, one derives expressions for the unit tangent and normal vectors

$$\vec{b} = \frac{\vec{B}}{|B|}, \vec{n} = \frac{\nabla\psi}{|\nabla\psi|}$$

$$\kappa_n = \vec{n}\cdot(\vec{b}\cdot\nabla)\vec{b},$$

$$\kappa_g = \vec{b}\times\vec{n}\cdot(\vec{b}\cdot\nabla)\vec{b},$$

$$\tau_n = -\vec{n}\cdot(\vec{b}\cdot\nabla)\vec{b}\times\vec{n}$$

  - κ_n = normal curvature
  - κ_g = geodesic curvature
  - τ_n = normal torsion ~ the `twist' of a field line

Frenet’s equations describe the “evolution” properties of the basis set

- “Evolution” properties for the orthonormal basis of unit vectors, **b**, **n**, **b×n** satisfy a variant of Frenet’s formulae

$$(\vec{b}\cdot\nabla)\vec{b} = \kappa_n \vec{n} + \kappa_g \vec{b}\times\vec{n},$$

$$(\vec{b}\cdot\nabla)\vec{n} = -\kappa_n \vec{b} + \tau_n \vec{b}\times\vec{n},$$

$$(\vec{b}\cdot\nabla)(\vec{b}\times\vec{n}) = -\tau_n \vec{n} - \kappa_g \vec{b}$$

- The curvature and torsion vectors describe the fundamental geometric properties of the field line.
  - ‘Bad’ curvature accesses the pressure gradient free-energy for instability drive.

The local shear is influenced by magnetic geometry and plasma profiles

- The local shear enters in describing the stabilizing role of magnetic field line bending energy in MHD stability theory.
- Normal torsion τ_n and parallel plasma currents are related to local shear by an identity.

$$s = (\vec{b} \times \vec{n}) \cdot \nabla \times (\vec{b} \times \vec{n})$$

$$s = \frac{\vec{J} \cdot \vec{B}}{B^2} - 2\tau_n$$

- The average shear is calculated from a flux surface average of the identity

$$\frac{d\iota}{d\psi} = < \frac{B^2 \hat{V}'}{|\nabla\psi|^2} s > = < \frac{\vec{J} \cdot \vec{B}}{|\nabla\psi|^2} \hat{V}' > - < 2\tau_n \frac{B^2 \hat{V}'}{|\nabla\psi|^2} >$$

Both magnetic geometry and profiles affect the local shear properties

- Local shear $s = J_{\|}/B - 2\tau_n$
- Parallel current = net current + Pfirsch-Schluter currents

$$\frac{J_{\|}}{B} = \sigma_o + \frac{dp}{d\psi}\lambda = \frac{<\vec{J}\cdot\vec{B}>}{<B^2>} + \frac{dp}{d\psi}\lambda,$$

$$\vec{B}\cdot\nabla\lambda = 2\frac{|\nabla\psi|}{B}\kappa_g$$

- Geometry
  - κ_g
  - τ_n
- Profiles
  - $dp/d\psi$
  - $d\iota/d\psi$
    - Net current needs to be consistent with averaged shear equation

$$\frac{d\iota}{d\psi} = <\frac{B^2\hat{V}'}{|\nabla\psi|^2}s> = <\frac{\vec{J}\cdot\vec{B}}{|\nabla\psi|^2}\hat{V}'> - <2\tau_n\frac{B^2\hat{V}'}{|\nabla\psi|^2}>$$

The Mercier criterion has a simple form when "geometry" is distinguished from profiles

- The Mercier criterion for localized interchange modes is given by

$$D_I = \frac{p'}{\iota'^2} W_M$$

- Here, W_M is a purely geometric object (Greene and Chance, NF '81; Greene , PPCF '97).

$$W_M = V'^2 < \frac{B^2}{|\nabla\psi|^2} > [\{\frac{2\kappa_n |\nabla\psi|}{B^2}\} - \{\lambda\}\{2\tau_n\} + \{2\tau_n\lambda\}]$$

where

$$\{f\} = \frac{\oint \frac{dl}{B}\frac{B^2}{|\nabla\psi|^2} f}{\oint \frac{dl}{B}\frac{B^2}{|\nabla\psi|^2}} \qquad \vec{B}\cdot\nabla\lambda = 2\kappa_g \frac{|\nabla\psi|}{B}$$

- The stability boundaries (D_I = 1/4) is a parabola in profile parameter space.

$$-p' = \frac{\iota'^2}{4(-W_M)}$$

A local helical axis equilibrium can be constructed

- A model quasi-helically symmetric configuration,
  - $B \sim B(N\zeta - \theta) \rightarrow \kappa_n, \kappa_g$ dominated by a single harmonic
    $\eta = N\zeta - \theta$
- Local helical axis equilibria specified in cylindrical coordinates $[R, \phi = -\zeta, Z]$

$$R = R_o + \rho_o \cos\theta + \Delta\cos(N\zeta) + \frac{2R_o\rho_o}{N^2\Delta}\sin(N\zeta)\sin\theta,$$

$$Z = \rho_o \sin\theta + \Delta\sin(N\zeta) - \frac{2R_o\rho_o}{N^2\Delta}\sin(N\zeta)\cos\theta$$

- In the limit $N^2\Delta/R_o >> 1 > N\Delta/R_o$

$$\kappa_n \approx -\frac{N^2\Delta}{R_o^2}\cos(N\zeta - \theta), \kappa_g \approx -\frac{N^2\Delta}{R_o^2}\sin(N\zeta - \theta),$$

$$\tau_n \approx -\frac{2}{N\Delta}\cos(N\zeta) + \frac{\iota_o}{R_o} - \frac{N^3\Delta^2}{R_o^3}\cos^2(N\zeta - \theta)$$

- The curvature vector is dominated by a single harmonic, the normal torsion contains a symmetry breaking harmonic

Ideal MHD stability boundaries can be obtained as functions of profiles and shaping

- Ballooning equation for local axis equilibria, $\eta = N\zeta - \theta$ labels points along the field line,

$$\frac{\partial}{\partial\eta}(1+\Lambda^2)\frac{\partial\xi}{\partial\eta} + \alpha[\cos(\eta) + \Lambda\sin(\eta)]\xi = -\Omega^2(1+\Lambda^2)\xi$$

- Integrated local shear (χ = field line label)

$$\Lambda(\eta,\chi) = \int_{\eta_k}^{\eta} d\eta'[s - \alpha\cos(\eta') + \tau_o\cos(2\eta') + \delta\cos(k\eta' + k\chi)]$$

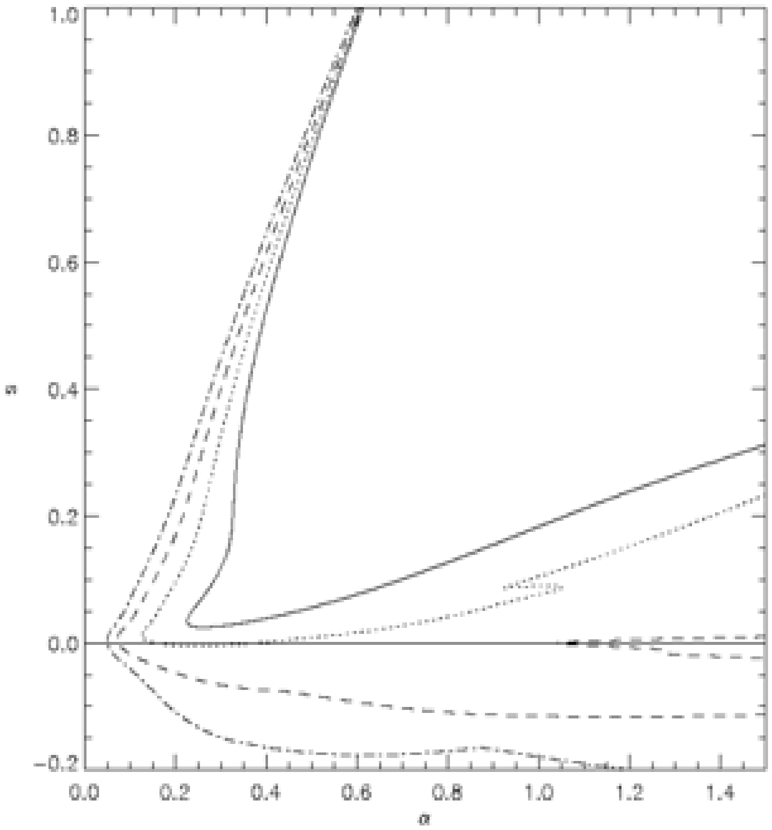
- 2 profile quantities $\alpha \sim -\frac{dp}{d\psi}, s \sim \frac{d\iota}{d\psi}$

- Geometric quantities

$$\delta = \frac{4R_o}{N(N-\iota_o)\Delta}, k = \frac{N}{N-\iota_o}, \tau_o = \frac{N^2\Delta^2}{R_o^2}$$

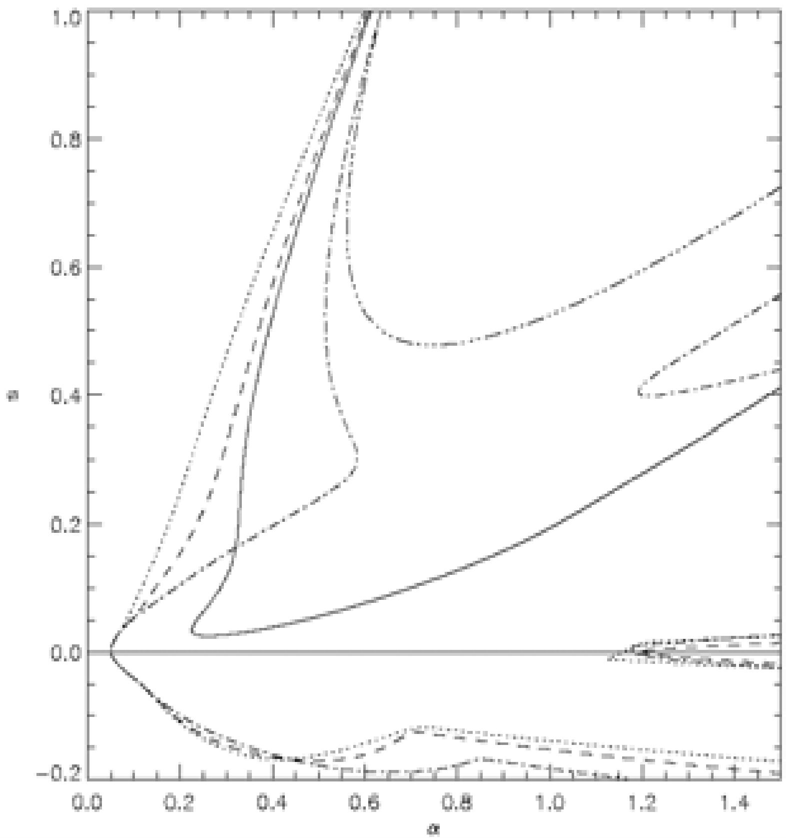
- For τ_o, $\delta \rightarrow 0$, ballooning equation is equivalent to axisymmetric circular equilibrium corrected for the change in the "connection length" $qR \rightarrow R_o/(\iota_o - N)$

Symmetry breaking contributions to the local shear reduce first stability boundaries and eliminate the second stable region



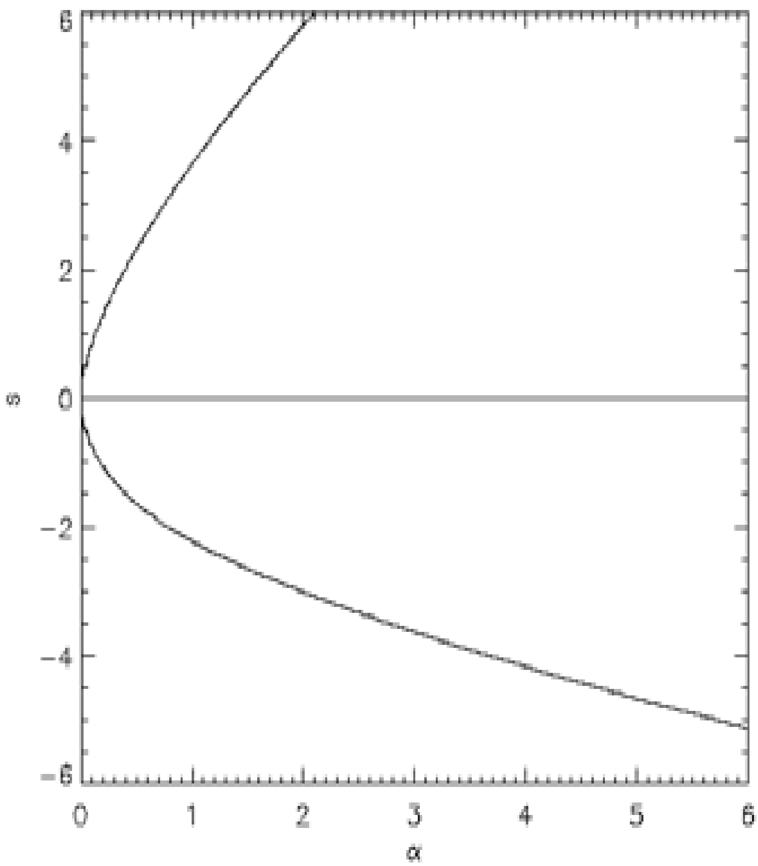
- Ideal stability boundaries for $\delta = 0, 0.15, 0.3, 0.45$ ($\tau_o = 0$, $k = \pi^2/8$).
- Solid line is the perfectly symmetric case - equivalent to the s-α tokamak equilibra
- With increasing symmetry breaking in the local shear-stability degrades.
- Instability is indicated if a single field line at a single value of η_k is unstable.

The ballooning eigenvalues are field line dependent in the small average shear region



- Ballooning eigenvalue $\Omega^2 = \Omega^2(\chi)$ with $\delta = 0.45$, $\chi = 0$, 0.85, 1.7, 2.55 (different field lines on the same magnetic surface).
- Local instability criterion is violated on a subset of the magnetic surface's field lines
- Unique property of 3-D.
  - For a particular value of s and α, mix of 'unstable' and 'stable' field lines on every surface.

Characteristics of the analytic model are reproduced even when some of asymptotic relations are relaxed



- Local equilibrium defined by

$$R = R_o + \rho_o \cos\theta + \Delta\cos(N\zeta) + \frac{2R_o\rho_o}{N^2\Delta}\sin(N\zeta)\sin\theta,$$

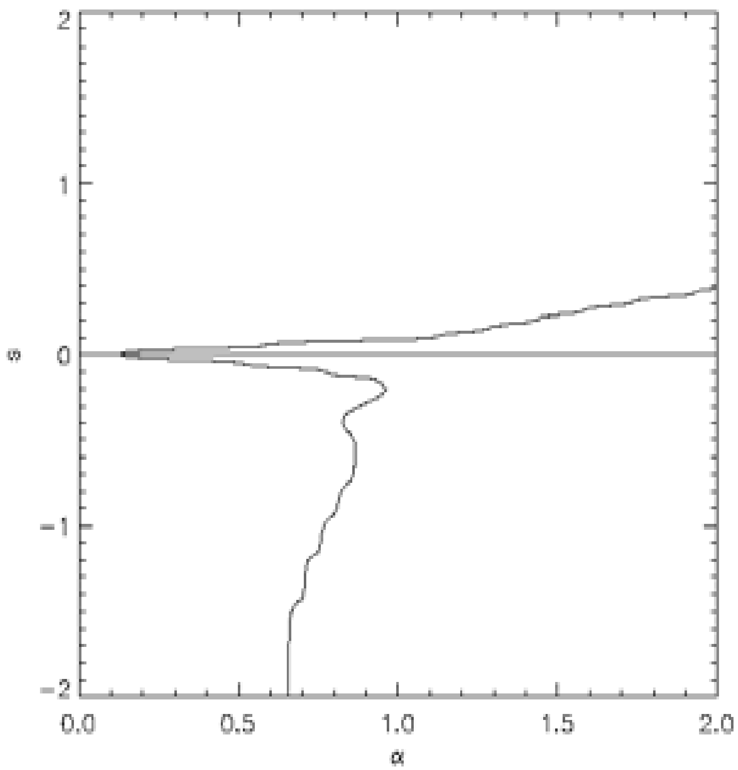
$$Z = \rho_o \sin\theta + \Delta\sin(N\zeta) - \frac{2R_o\rho_o}{N^2\Delta}\sin(N\zeta)\cos\theta$$

  with N = 10, $\Delta/R = 0.01875$, $\rho_o/R_o = 0.1$

  – No second stable region

  – Field line dependent eigenvalues

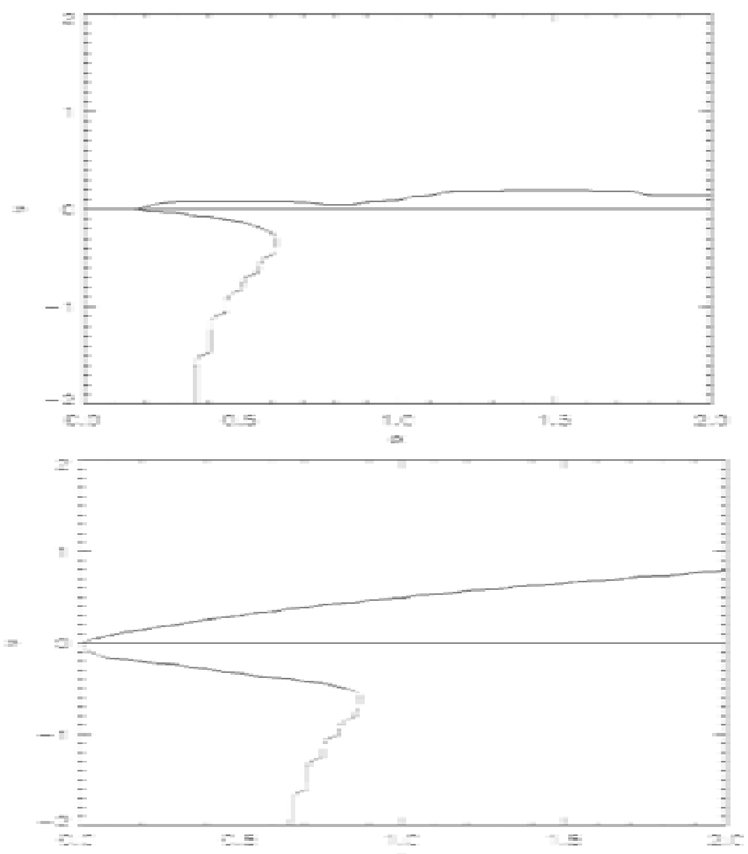
The HSX standard case shows no second stable region



- The HSX standard configuration with N = 4, |B| ~ B(ψ)[1 - $\varepsilon_h$cos(Nζ-θ)],

  R_o = 1.2 m, <a> = 0.15 m, average helical excursion of axis = 0.2m. Typical transform, ι_{axis} ≅1.05, Δι ≅0.07
- Characteristic ballooning instability violated at β ≅0.7%

Other magnetic configurations available to HSX do not seem to dramatically alter the ballooning stability properties



- HSX has a variety of configurations, including one with a deep magnetic well, $\iota_{axis} \cong 1.14$, $\Delta\iota \cong 0.1$, and one with mirror terms, $\iota_{axis} \cong 1.05$, $\Delta\iota \cong 0.07$.
- While quantitative predictions for ballooning stability onset are different, the structure of the stability boundary is not radically different from the standard quasi-helical case.

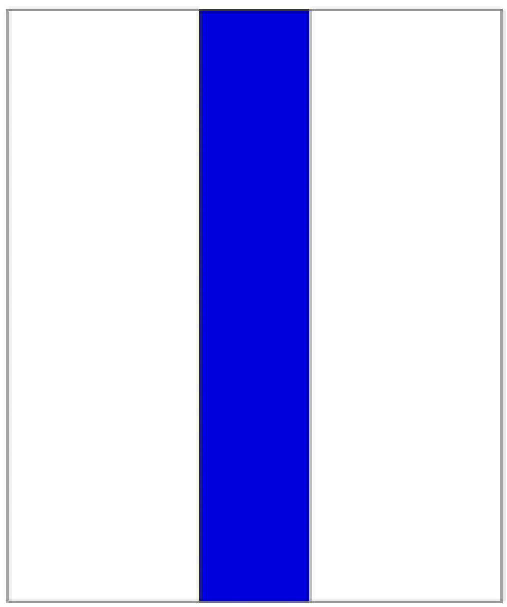
A geometric interpretation of the results

$\theta \rightarrow$

Bad curvature in blue

ζ

$\Downarrow$



- For a tokamak, curvature is unfavorable near $\theta = 0$.

- Ballooning instability ensues when regions of small local shear intersect regions of bad curvature

The pressure gradient affects the local shear

Local shear for a tokamak

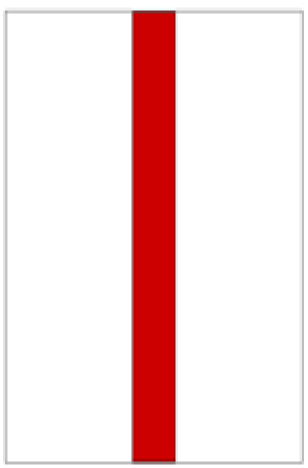
$\cong$ s - α cos(θ)

- At small pressure gradient, local shear $\neq$ 0.

$\theta \rightarrow$ Local shear with s ~ α

ζ

$\Downarrow$

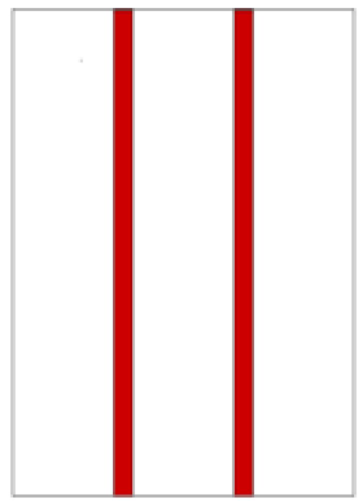


- At moderate pressure gradient, s ~ α, instability
- At high α, local shear ~ 0 point migrates away from θ = 0

$\theta \rightarrow$ Local shear at higher α

ζ

$\Downarrow$



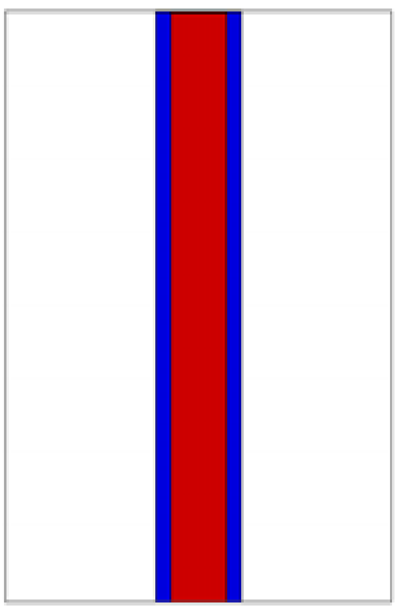
Overlap of bad curvature region with regions of small local shear indicate instability

- Overlap of bad curvature and small local shear indicate instability

$\theta \rightarrow$

ζ

$\Downarrow$

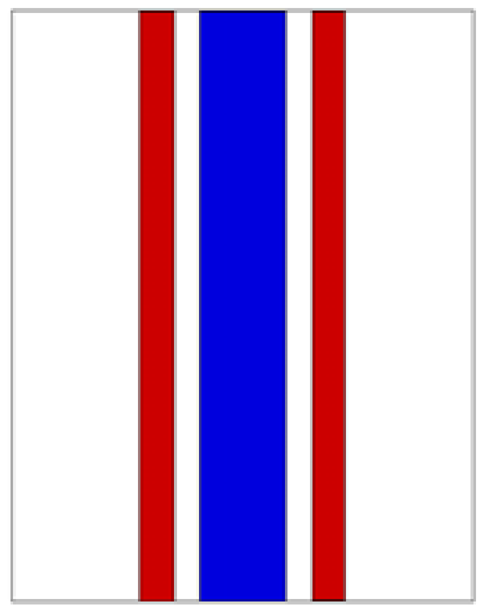


- Second stability occurs at sufficiently high Grad-p (Greene and Chance, NF ‘81)

$\theta \rightarrow$

ζ

$\Downarrow$



Let's consider the stability properties of a quasi-helically symmetric configuration

- Consider a quasisymmetric configuration with curvature
  - $\kappa_n \sim \cos(N\zeta - \theta)$
- More general 3-D configurations would have more structure in |B| and hence affects detailed quantitative comparions, but this simple example provides a useful illustration,

$\theta \rightarrow$
ζ
$\Downarrow$

The local shear of 3-D configurations do not generally have the symmetry as the curvature

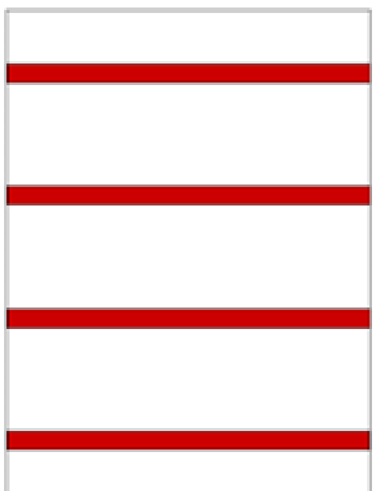
- Model local shear for quasi-helically symmetric device

  $\approx s - \alpha \cos(N\theta - \zeta) - \delta \cos(N\zeta)$

$\theta \rightarrow$ For $\alpha = 0$

ζ

$\Downarrow$

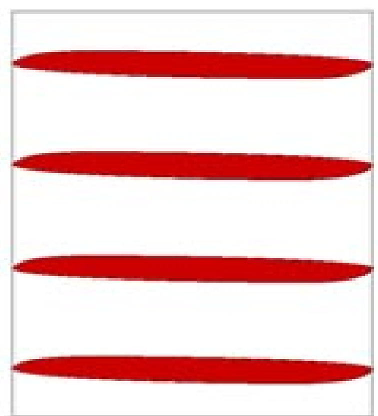


- At small α, the helical content of the local shear differs from the curvature
- At higher α, small local shear region deforms

$\theta \rightarrow$ For $\alpha \neq 0$

ζ

$\Downarrow$



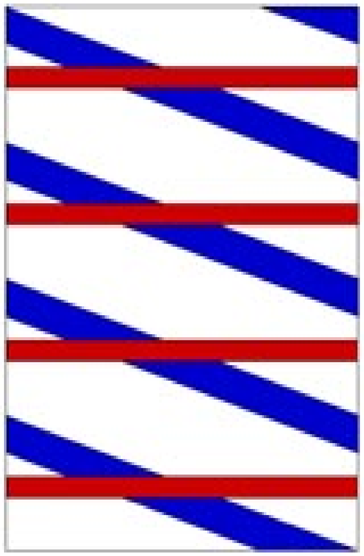
The lack of a continuous symmetry adversely impacts the local stability properties

Overlap of bad curvature and small local shear regions,

$\Theta \rightarrow$

Z

$\Downarrow$



- Overlap occurs at $\alpha = 0$
- Highly field-line dependent
- At small α, zeroes of the local shear occur as functions of an angle that is incommensurate with the $|B|$ symmetry.
- At higher α, the small local shear region deforms, but in general it is difficult to find a case where local shear nulls avoid bad curvature regions.

The stability properties of 3-D systems differ from symmetric systems in a number of ways

- Local criteria "predicts" instability at a critical pressure gradient.
  - Criteria violated initially on a single field line on a particular magnetic surface.
  - Local criteria violation = β–limit?
    - Highly localized
      - Easily FLR stabilized?
      - Small nonlinear saturated amplitude?
      - Global mode structure? (non-integrability of WKB ray equations - Dewar and Glasser, PF '83).
  - Reliance on local stability criterion = over constraining experimental designs?

Summary

- A technique is introduced to manipulate profile and shaping properties of 3-D configurations. $\rightarrow$ “Local 3-D equilibria” $\rightarrow$ Allows one to map out regions of stability for localized modes as functions of 3-D shaping and profile effects.
  - Also applicable to transport, microinstability calculations
- An example local helical axis model is introduced to emphasize the technique. $\rightarrow$ Allows one to adjust analytically the strength of symmetry breaking terms in the local shear.
- Ideal MHD ballooning stability boundaries are calculated for the helically symmetric equilibria.
  - Sensitivity to symmetry breaking terms in the local shear,
  - Deterioration of the first stability boundary, reduction and/or elimination of second stability regimes.
  - Field line dependent eigenvalues.