Stellarator Coil Optimization by Targeting the Plasma Configuration

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Modern stellarators are designed by choosing the shape of the plasma to optimize its physics properties. Unfortunately practical magnetic field coils cannot precisely support a plasma that has the shape prescribed by the optimization. A method will be given for optimizing plasmas so the only plasma shapes that are considered are the ones that are consistent with efficient, well-separated coils. The method provides a clear prescription for finding the minimal number of independent coils needed for the flexibility to produce many important plasma configurations. The number of free parameters that exist for optimizing stellarator configurations will be determined. About thirty parameters in stellarator design are of comparable difficulty for coils to produce as the four shape parameters that are used in defining tokamaks through squareness.

I. Introduction

The physics properties of a toroidal plasma are uniquely determined by the profiles of net plasma current and pressure, the toroidal flux in the plasma, and the plasma shape.[1] That is, the shape of the outermost surface of the plasma on which the normal component of the magnetic field, $\vec{B}\cdot\hat{n}$, must vanish. Jürgen Nührenberg[2,3] used the uniqueness to find stellarators with optimal physics properties by varying the shape of the plasma surface. This procedure is the basis of modern stellarator design.

Nührenberg's procedure maximizes a target function that depends on a set of parameters that define the plasma surface. The location of the outermost plasma surface, $\vec{x}(\theta,\varphi)$, is conventionally described using (R,φ,Z) cylindrical coordinates,

$$\vec{x}(\theta,\varphi) = R(\theta,\varphi)\hat{R}(\varphi) + Z(\theta,\varphi)\hat{Z}. \tag{1}$$

The functions of two angles, $R(\theta,\varphi)$ and $Z(\theta,\varphi)$, are defined by a set of coefficients s_i, which can be written as a matrix vector $\vec{s}$. For example, the s_i may be the coefficients of Fourier series. The physics properties of a stellarator plasma, with given pressure and current profiles, are then determined by $\vec{s}$. The magnitude of the magnetic field is normalized by the toroidal magnetic flux within the plasma boundary. In Nührenberg's procedure, a target function $T(\vec{s})$ is defined, which contains information about the neoclassical transport and magnetohydrodynamic properties of the plasma. The shape coefficients $\vec{s}$ are varied until the target function is maximized.

In modern stellarator design, coils to support an optimized stellarator plasma are found by following a procedure given by Peter Merkel.[4] The coils have two functions. They (1) produce the net toroidal magnetic flux in the plasma, and (2) insure the normal component of the magnetic field on the plasma surface is zero. As Merkel observed in his original paper on coil design,[4] coils with a non-zero separation from the plasma cannot precisely accomplish the second of these tasks for a defined plasma shape. The mathematical reasons will be discussed in Section (II). The method adopted by Merkel for dealing with this problem is to design the coils to minimize the mean-squared normal magnetic field on the desired plasma surface,

$$\mathcal{E}^2 \equiv \oint_{plasma} \left(\vec{B}\cdot\hat{n}\right)^2 da. \tag{2}$$

Once coils are designed by this method, further optimizations can be carried out by varying the shapes of the coils.[5]

Merkel's basic strategy of minimizing $\mathcal{E}^2$ treats all distributions of the normal magnetic field on the plasma surface with an equal weight. However, the physics properties of a stellarator can be far more sensitive to some normal field distributions than others. This observation leads to two different strategies for choosing the coils. In the first strategy, which was described in a recent paper,[6] the physics optimization defines not only a plasma shape but also an acceptable tolerance on the various distributions of normal magnetic field. Coils are then optimized within these constraints. A distribution of the normal magnetic field that must be retained to minimize $\mathcal{E}^2$ might have little effect on the physics properties of the plasma. If that normal field distribution is associated with a magnetic field that decays rapidly away from coils, inefficient and nearby coils are needlessly required. On the other hand if plasma properties are sensitive to a particular field distribution, the importance of that distribution may not be adequately represented in a minimization of $\mathcal{E}^2$. In the second strategy, which is described in Section (III), the set of magnetic field distributions that have a slow spatial decay away from coils is established. These are the only field distributions that are consistent with efficient and distant coils. The plasma optimizations are carried out using the amplitudes of these field distributions as independent parameters. Walter Dommaschk has given scalar potentials for a set of slowly decaying fields,[7] which could be used in the optimization. The sets of slowly decaying fields described in Section (III) obey completeness and orthogonality relations that are not simply obtained for the Dommaschk potentials. In effect, the first strategy optimizes the plasma and then checks for the implications for the coils. The second optimizes the coils and then checks for the implications for the plasma.

The codes that have been written for physics optimization can be modified to use the set of magnetic field distributions that have a slow spatial decay. The physics target function is augmented by a penalty function that measures the difference between the normal magnetic field that a certain plasma shape requires and the normal field that the set of magnetic field distributions can produce. If the constant that appears in front of this penalty function, c_p, is small, the physics optimization is of the traditional form. If c_p is

made large, the degradation in the physics properties associated with efficient and distant coils is explicitly determined. By gradually making c_p larger, one can determine which missing field distribution would have the largest effect on the physics optimization.

The utility of a stellarator experiment is largely determined by the ability of its coils to support a large number of interesting plasma equilibria. Emphasis should be placed on interesting equilibria. Any coil set offers some flexibility through variations in the currents in the coils. However careful design is required if variations in the coil currents are to produce an interesting range of equilibria. In the Wendelstein 7-X stellarator, for example, flexibility in interesting equilibria was achieved[8] by the use of a second set of coils to allow variation in rotational transform while maintaining optimized plasma configurations. In Section (IV) techniques will be discussed that allow coils to be designed, so the widest possible range of interesting configurations can be obtained by variations in the coil currents.

The procedure for designing coils that will be given in this paper uses a control surface, which is separated from all of the desired plasma configurations by at least the minimum coil/plasma separation. The magnetic field distributions are determined by the use of a complete set of distributions of surface current on the control surface. These distributions are ordered by the rate of spatial decay of their magnetic field in the region enclosed by the control surface. For typical stellarators approximately thirty distributions have a spatial decay no faster than the field distribution that determines squareness in a tokamak. These distributions are the ones that are available to optimize plasmas that are consistent with reasonably distant coils. The current distribution on the control surface can be divided into three parts: (i) a primary current distribution, which produces a typical plasma configuration, (ii) a set of distributions required for flexibility, and (iii) a set of tolerances on the accuracy with which a complete set of current distributions must be controlled. The flexibility requirements are based on the consideration of a large number of plasma equilibria, such as different pressure and current profiles, and the determination by Singular Value Decomposition (SVD) methods[9] of the most important current distributions for flexibility. These current distributions can be used as the basis of the coil design, or they can be converted into normal magnetic field distributions on the control surface. The formulation using the normal magnetic field distributions allows a simple specification of the physics requirements on the coils. With a simple specification of the requirements, coil designers can use their full creativity in finding the cheapest set of coils that offer adequate access to the plasma.

A number of persons have noted that a symmetric $1/R$ toroidal field produces a normal magnetic field with high harmonics when interacting with a complicated plasma boundary. They have viewed this as proof that a slowly decaying field can produce the high harmonic normal fields that are desired for stellarator optimization. This view confuses the problem and the solution as is illustrated by giving the surface of a tokamak a toroidal ripple. Few will doubt that if one wishes to produce a mode number n toroidal ripple in a tokamak surface that a field that scales roughly as R^n is required. Nonetheless, an axisymmetric toroidal field dotted into the normal to the rippled surface has an nth toroidal harmonic. A tokamak surface with sufficiently high order toroidal ripple is inconsistent with distant coils.

II. Mathematical Issues

The design of stellarator coils raises a number of interesting and important mathematical issues. Two of these will be discussed in this section. The first issue is why it is an ill-posed mathematical problem to specify the normal magnetic field on the plasma surface that a set of coils must produce. The second issue is the determination of the choices that exist in the specification of a unique magnetic field in the region enclosed the coils. This section can be skipped by readers more interested in the coil design method than in the associated mathematical issues.

In the paper that originated modern coil design,[4] Peter Merkel noted that the problem of the continuation of a magnetic field into a vacuum region is not well posed and that singularities tend to arise. In 1917 Jacques Hadamard[10] gave the example of this type of problem that appears in textbooks on partial differential equations. However, the problem in coil design is somewhat different than that given by Hadamard. Hadamard showed that Cauchy boundary conditions (specification of both the function, $\phi(\vec{x})$, and the normal derivative of the function, $\hat{n}\cdot\vec{\nabla}\phi$) yield ill-behaved solutions when applied to Laplace's equation, $\nabla^2\phi = 0$. However in standard coil design, Cauchy boundary conditions are never used. The boundary conditions that are used are the function (Dirichlet conditions) or the normal derivative of the function (Neumann conditions). Indeed to obtain a unique solution to Laplace's equation, one must apply either Dirichlet or Neumann boundary conditions. See the discussion of Equation (9) below.

To illustrate the type of problem faced in coil design, consider $\nabla^2\phi = 0$ in (r,θ) polar coordinates. The well-behaved (regular) solution in the region $r \le b$ is

$$\phi(r,\theta) = \sum_{m=1}^{\infty} c_m \left(\frac{r}{b}\right)^m \exp(-im\theta) \tag{3}$$

with the c_m constants. These constants can be determined by specifying either the function $\phi(a,\theta)$ (Dirichlet conditions) or its normal derivative $\left(\partial\phi/\partial r\right)_{r=a}$ (Neumann conditions) at $r=a$. However, either specification generally makes Laplace's equation ill-posed (yields singular results) when $a < b$. To demonstrate this, consider the function

$$\phi(a,\theta) = \frac{1}{i}\,\mathrm{Imag}\left[\frac{1}{1-\alpha \mathrm{e}^{\mathrm{i}\theta}}\right] = \frac{\alpha\sin\theta}{1+\alpha^2-2\alpha\cos\theta} = \sum_{m=1}^{\infty}\alpha^m \sin(m\theta), \tag{4}$$

which is a well-behaved analytic function for $\alpha < 1$. The solution at $r=b$ is

$$\phi(b,\theta) = \sum_{m=1}^{\infty}\left(\alpha\frac{b}{a}\right)^m \sin(m\theta), \tag{5}$$

which is an exponentially divergent Fourier series when $b > a/\alpha$. The divergence of the Fourier series for $\phi(b,\theta)$ is the generic result when $\phi(a,\theta)$ is an analytic function of θ. Generic analytic functions have singularities (poles) in the complex θ-plane. Let the closest pole above the real axis be located at $\theta = \Theta + i\Gamma$ with $\phi(a,\theta) \to c_r/\left(\theta - (\Theta + i\Gamma)\right)$ near that pole. The Fourier transform, $2\pi c_m \equiv \oint \phi(a,\theta)\exp(im\theta)d\theta$, can be calculated by standard contour integration methods to yield $c_m \to \{ic_r \exp(im\Theta)\}\exp(-m\Gamma)$ for large m. In other words, the Fourier series of a generic analytic function converges exponentially, as $\exp(-m\Gamma)$. The example of Equation (4) has an exponentially convergent series with $\alpha = e^{-\Gamma}$. An analytic expression for the normal magnetic field on the plasma surface (which is actually a Neumann condition) generically yields a singular current for a sufficiently separated coil surface. However in the cylindrical example, a generic analytic function can be approximated with exponential accuracy by a function for which the solution at $r=b$ increases only algebraically, as $(b/a)^M$ with M an integer. That is, the

Fourier series for an analytic function can be truncated at $m=M$, which yields a analytic function that differs from the original one by an exponentially small, $\exp(-M\Gamma)$, function of θ.

The problem of going from the normal field specified on the plasma surface to the coils is ill-posed. In the example of the last paragraph, a function that approximates the required normal field with exponential accuracy, $\exp(-M\Gamma)$, is consistent with coils that are separated from the plasma by an arbitrarily large distance. However, the required currents can become very large, increasing as $(b/a)^M$ with M the largest mode number that is retained. Efficient, but distant, coils require M be small. However, M must be sufficiently large to achieve the desired plasma properties. It is the achievement of an optimal balance desirable coils and desirable plasmas that is the goal of both the method explained in this paper and in the method of the previous publication.[6]

The magnetic field that is required for supporting a plasma can be specified in several ways. The choices in the quantities that can be specified in order to define a unique magnetic field will now be explored.

Electrodynamics is based on linear equations, so the magnetic field in an experiment is the unique sum of the field due to the coil currents and the field due to the plasma currents, $\vec{\nabla}\times\vec{B}_{pl}=\mu_o\vec{j}_{pl}$ with $\vec{B}_{pl}(\vec{x})\to 0$ as $|\vec{x}|\to\infty$. To simplify notation, the field due to the coil currents will be denoted by $\vec{B}$, so the complete magnetic field is $\vec{B}_{pl}+\vec{B}$. By the control surface is meant a toroidal surface that lies outside of all plasmas a stellarator is designed to produce with a spatial separation from these plasmas which is no less than the minimum allowed coil/plasma separation. The control surface is assumed to lie between the plasmas and all coils. In the region enclosed by the control surface, the field due to the coils is $\vec{B}=\vec{\nabla}\phi(\vec{x})$ with $\nabla^2\phi=0$.

In the region bounded by the control surface, the magnetic field $\vec{B}$ obeys a simple integral relation, Equation (8), which defines the uniqueness requirements. In this region the scalar potential of the field cannot be single valued for coils that have a net poloidal current (the current through the hole of the torus). That current is given by $\mu_0 G=\oint\vec{B}\cdot(\partial\vec{x}/\partial\varphi)d\varphi$. The scalar potential for the magnetic field has the form

$\phi = (\mu_0 / 2\pi) G\varphi + \tilde{\phi}$ with $\tilde{\phi}$ a single-valued function of position. Within the region enclosed by the control surface

$$\int B^2 d^3x = \int \left\{ \vec{\nabla} \cdot \left(\phi \vec{\nabla} \phi \right) - \phi \nabla^2 \phi \right\} d^3x = \oint_{control} \tilde{\phi} \vec{B} \cdot d\vec{a} + \frac{\mu_0 G}{2\pi} \int \vec{\nabla} \cdot (\varphi \vec{B}) d^3x . \tag{6}$$

Since the magnetic field is divergence free, $\vec{\nabla} \cdot (\varphi \vec{B}) = \vec{B} \cdot \vec{\nabla} \varphi$, and

$$\int \vec{B} \cdot \vec{\nabla} \varphi d^3x = 2\pi \psi_t \tag{7}$$

with ψ_t the toroidal magnetic flux enclosed by the control surface. Therefore,

$$\int_{enclosed} B^2 d^3x = \mu_0 G \psi_t + \oint_{control} \tilde{\phi} \vec{B} \cdot d\vec{a} . \tag{8}$$

This equation allows one to find the conditions required for a unique magnetic field within the region enclosed by the control surface. Let $\delta \vec{B} \equiv \vec{B}_1 - \vec{B}_2$ be the difference between two magnetic field distributions in the region enclosed by the control surface. Then, a derivation identical to that of Equation (8) implies

$$\int_{enclosed} \left(\delta \vec{B} \right)^2 d^3x = \mu_0 (\delta G)(\delta \psi_t) + \oint_{control} \left(\delta \tilde{\phi} \right) \left(\delta \vec{B} \right) \cdot d\vec{a} . \tag{9}$$

The two fields must be identical, $\delta \vec{B} \equiv 0$, if the right hand side of this equation vanishes. Two conditions must be satisfied to make the two fields identical. First, either the poloidal currents G must be identical or the toroidal fluxes ψ_t must be identical. Second, either the single-valued potentials $\tilde{\phi}$ must be identical (Dirichlet conditions), or the normal components of the fields $\vec{B} \cdot \hat{n} = \hat{n} \cdot \vec{\nabla} \phi$ must be identical (Neumann conditions).

A uniqueness theorem can also be obtained in terms of the surface current on the control surface that would be required to produce the magnetic field due to the coils. The

current density on the control surface $\vec{j}$ must be divergence free and lie in a toroidal surface. These two conditions imply the form

$$\vec{j} = \vec{\nabla} \times \left\{ \left(\kappa(\theta,\varphi) + \frac{G}{2\pi}\varphi - \frac{J}{2\pi}\theta \right) \delta(r-b) \vec{\nabla} r \right\}. \tag{10}$$

with (r, θ,φ) any toroidal coordinate system in which $r=b$ defines the control surface. The net poloidal current in the control surface is the constant G, and the net toroidal current is $J = \left(\oint \vec{B} \cdot (\partial \vec{x} / \partial \theta) d\theta \right) / \mu_0$. The function $\kappa(\theta,\varphi)$ is called the current potential, has units of Amperes, and defines the distribution of the surface current. The magnetic field

$$\vec{B} = \vec{\nabla}\phi + \mu_0 \left(\kappa(\theta,\varphi) + \frac{G}{2\pi}\varphi - \frac{J}{2\pi}\theta \right) \delta(r-b) \vec{\nabla} r. \tag{11}$$

satisfies Ampere's Law, $\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$. Except on the control surface, the magnetic field satisfies $\vec{B} = \vec{\nabla}\phi$ with $\nabla^2 \phi = 0$, and throughout all of space the magnetic field is non-infinite and divergence-free. Integrating $\vec{B} \cdot \partial \vec{x} / \partial r$ an infinitesimal radial distance across the control surface, one finds that the jump in the potential across the control surface (outside minus inside) is

$$\frac{1}{\mu_0}[\phi] = -\kappa - \frac{G}{2\pi}\varphi + \frac{J}{2\pi}\theta. \tag{12}$$

The net toroidal current flowing in the control surface J is given by the poloidal loop integral $J = \left(\oint \vec{B} \cdot (\partial \vec{x} / \partial \theta) d\theta \right) / \mu_0$, so the magnetic potential must have a multivalued term, $(\mu_o J / 2\pi)\theta$, in the region $r > b$. If $\vec{B} \cdot \vec{\nabla}\phi$ is integrated over all of space, except the region occupied by the coil surface, one finds

$$\int \vec{B}^2 d^3x = \int \vec{\nabla} \cdot (\phi \vec{B}) d^3x = -\oint_{control} \left[\tilde{\phi} \right] \vec{B} \cdot d\vec{a} + \mu_0 G \psi_t + \mu_0 J \psi_p. \tag{13}$$

The condition $\left[\vec{B}\cdot\hat{n}\right]=0$, which follows from $\vec{\nabla}\cdot\vec{B}=0$, was used. The poloidal flux (the flux through the hole in the torus) is

$$\psi_p \equiv \frac{1}{2\pi}\int_{exterior}\vec{B}\cdot\nabla\theta d^3x \ . \tag{14}$$

Equation (12) implies the single-valued part of the magnetic potential obeys $\left[\tilde{\phi}\right]=-\mu_0\kappa$, so

$$\int_{all_space}\left(\vec{B}^2/\mu_0\right)d^3x = \oint_{control}\kappa\vec{B}\cdot d\vec{a}+G\psi_t+J\psi_p . \tag{15}$$

The net toroidal current J has no direct effect on the field in the region enclosed by the control surface, and the net poloidal current G has no direct effect on the field outside of the control surface. If currents lie on a number of surfaces, Equation (15) can be trivially generalized by letting the right hand side be a sum of terms, each of which is like the right hand side of Equation (15).

The magnetic field can be made unique in the region enclosed by the control surface by specifying either the poloidal current G or the toroidal flux ψ_t together with either the current potential κ on the control surface, Equation (15), or the normal field $\vec{B}\cdot\hat{n}$ on the control surface, Equation (8).

III. Optimization Strategy

The central idea is to optimize stellarator configurations using the distributions of current $\vec{j}$ on the control surface that produce magnetic fields that have a slow spatial decay. An arbitrary current $\vec{j}$ on the control can be described by a current potential $\kappa(\theta,\varphi)$ together with the net toroidal J and poloidal G currents flowing on the surface, Equation (10). The current potential on the control surface can be expanded in a complete set of orthonormal functions, $\kappa(\theta,\varphi)=\sum I_i g_i(\theta,\varphi)$, which gives an infinite number of parameters, the I_i, with which to optimize stellarator configurations. However, if the othonormal functions are chosen appropriately only a few, N_o, are associated with magnetic fields that have a sufficiently slow spatial decay to be useful for optimizing a

stellarator that is to be consistent with efficient and distant coils. As will be discussed in Section (V) the number N_o is related to the number of Fourier terms of the form $\sin(m\theta - n\varphi)$ that have $m^2 + n^2$ less than a critical value. By comparison with tokamak experience $m^2 + n^2 \leq 20$ is a reasonable constraint, which makes N_o=4 in a tokamak and N_o=34 in a stellarator. Although the parameter space available for stellarator optimization is limited, the available number of free parameters is not smaller than the number used in previous optimization studies—just different. If one wants stellarators that are consistent with distant coils, then it would appear reasonable to seek optima in the space of configurations that are consistent with distant coils. That is the gist of the present proposal. This strategy has the additional benefit of allowing the design of coils that have maximum flexibility for accommodating plasmas with different profiles and physics properties, Section (IV).

A simple idea for identifying an appropriate set of orthonormal functions for describing the current potential is to start with two arbitrary complete sets of orthonormal functions: $\phi_i(\theta,\varphi)$ on a surface that is typical of the location of plasmas that one wishes to produce and $\gamma_j(\theta,\varphi)$ on the control surface. A single term in the current potential on the control surface, $I_j\gamma_j(\theta,\varphi)$, produces a magnetic field $\vec{B}_j$ and a row of an inductance matrix,

$$L_{ij}I_j \equiv -\oint_{plasma} w_p\phi_i(\theta,\varphi)\vec{B}_j \cdot d\vec{a}. \tag{16}$$

A repeated index does not imply a sum. The function w_p is the weight that appears in the definition of orthonormality on the plasma surface, $\oint w_p\phi_i\phi_j da = \delta_{ij}$. The inductance matrix L_{ij} gives the strength of the normal magnetic field on the plasma surface that has the spatial distribution $\phi_i(\theta,\varphi)$ produced by a current potential in the control surface with spatial distribution $\gamma_j(\theta,\varphi)$. Neil Pomphrey introduced the use of Singular Value Decomposition theory for studying the inductance matrix of stellarator coils. The fundamental theorem of Singular Value Decomposition (SVD) theory[5] says an arbitrary real matrix such as $\overleftrightarrow{L}$ can be written as

$$\overleftrightarrow{L} = \overleftrightarrow{U}^T \cdot \overleftrightarrow{\ell} \cdot \overleftrightarrow{V} \tag{17}$$

with $\overleftrightarrow{U}$ and $\overleftrightarrow{V}$ orthogonal matrices and $\overleftrightarrow{\ell}$ a diagonal matrix. (An orthogonal matrix multiplied by its transpose is the unit matrix.) Let $\vec{\gamma}$ be a vector with components $\gamma_i(\theta,\varphi)$ and $\vec{g}$ a vector with components $g_i(\theta,\varphi)$ such that

$$\vec{g} = \overleftrightarrow{V} \cdot \vec{\gamma} \,. \tag{18}$$

The inductance, ℓ_i, associated with a function $g_i(\theta,\varphi)$ is a diagonal element of the inductance matrix $\overleftrightarrow{\ell}$ and gives the efficiency with which the current distribution $g_i(\theta,\varphi)$ produces normal magnetic field on the plasma surface. If the functions $g_i(\theta,\varphi)$ are ordered so the ones associated with larger inductances, the ℓ_i, come first, they form a useful set in which to expand the current potential. Only current potentials with a sufficiently large inductance should be considered in the stellarator optimization that emphasizes efficient coils.

The set of useful distributions of current potential in the control surface, the $g_i(\theta,\varphi)$, can be chosen in a number of ways. Two different sets of N_o useful distributions are essentially equally useful if they are coupled by non-singular matrix and each set uniquely determines the set of N_o distributions of magnetic field that have the slowest spatial decay. Not all sets of functions, g_i, have these properties. In one dimension any odd periodic function, $g(\theta) = g(\theta + 2\pi) = -g(-\theta)$, can be written as $g(\theta) = a\sin(\Theta)$ with a the maximum value of g and $\Theta(\theta) \equiv \arcsin(g/a)$. Consequently, a function with an arbitrarily rapid spatial variation can be written using a Fourier series that contains only an m=1 term. A choice that helps clarify fundamental properties of appropriate functions, g_i, and avoids the problems of Fourier series is the set of natural functions, which is discussed in Section (V).

A stellarator is optimized in the proposed method by picking N_o distributions of the current potential on the control surface that have sufficiently large inductances ℓ_i to produce a significant magnetic field on plasma surfaces. The optimization is done by augmenting the target function $T(\vec{s})$ of a Nührenberg-like optimization with a penalty function $P(\vec{s})$. The new target function is $T(\vec{s})$ - $P(\vec{s})$. Let the index j = 1 to N_o denote the distributions of current potential that are retained, and let $\vec{B}_j$ be their associated

magnetic fields. Linear algebra implies the inductance matrix, Eq. (16) defined by these distributions will have N_o non-zero eigenvalues and N_o associated functions $f_i(\theta,\varphi)$ on the plasma surface, $\vec{f} = \overleftrightarrow{U} \cdot \vec{\phi}$. The orthogonal matrix $\overleftrightarrow{U}$ is defined by Equation (17). Let $\vec{B}_o$ be the magnetic field due to all sources other than the current potential $\kappa = \sum I_i g_i(\theta,\varphi)$, and define fluxes $\Phi_i \equiv \oint w_p f_i \vec{B}_o \cdot d\vec{a}$. The penalty function can then be chosen to be

$$P(\vec{s}) = \left(1 - \frac{\sum_{i=1}^{N_o} \Phi_i^2}{\oint w_p (\vec{B}_o \cdot \hat{n})^2 da} \right) c_p \tag{19}$$

with c_p a constant. In stellarator optimization codes, the normal magnetic field on the surface of a plasma due to all other sources, $\vec{B}_o$, is determined by the plasma shape vector $\vec{s}$. Consequently, $\oint w_p (\vec{B}_o \cdot \hat{n})^2 da$ is a function of $\vec{s}$ as is each flux component, $\Phi_i \equiv \oint w_p f_i \vec{B}_o \cdot d\vec{a}$ on the plasma surface. If the constant c_p in Equation (19) is zero, then the new optimization is the same as the old. By increasing c_p during an optimization one can track the modification of a plasma configuration that is required to make it consistent with desirable coils, $c_p \to \infty$.

The number of parameters that are used to describe the shapes of the plasma surfaces in the VMEC equilibrium code,[11] which is at the heart of most stellarator optimizers, is in practice very limited. This limitation makes it subtle to constrain the surface shape to be consistent with distant coils. The number of magnetic field distributions that are consistent with distant coils, N_o, may even be greater than the number of shape parameters, $\vec{s}$. Nonetheless, the VMEC shape parameters require constraint to be consistent with distant coils. That is, the low-order VMEC shape parameters do not in themselves constrain the plasma shape to be consistent with distant coils. In the remainder of this section a method of finding the constraint on the shape parameters, $\vec{s}$, will be given, and the completeness of the shape parameterization will be defined.

Let $\vec{f} = \overleftrightarrow{U} \cdot \vec{\phi}$ be functions, $f_i(\theta,\varphi)$, that are ordered by the size of their associated inductances with fluxes, $\Phi_i \equiv \oint w_p f_i(\vec{B}_o \cdot \hat{n}) da$. If a small change is made in the shape, the fluxes are changed by $\delta\vec{\Phi} = \overleftrightarrow{\Omega} \cdot \delta\vec{s}$. Using Singular Value Decomposition one can write $\overleftrightarrow{\Omega} = \overleftrightarrow{Y}^T \cdot \overleftrightarrow{\omega} \cdot \overleftrightarrow{Z}$ with $\overleftrightarrow{\omega}$ diagonal and $\overleftrightarrow{Y}$ and $\overleftrightarrow{Z}$ orthogonal. The number of non-zero elements of $\overleftrightarrow{\omega}$ gives the number of effectively independent surface displacements N_s. (Changes in the shape parameters $\vec{s}$ can represent changes in the parameterization of the surface in (θ,φ) as well as changes in its shape.) Only effectively independent displacements will be considered, so $\overleftrightarrow{\omega}$ can be taken to have a well-defined inverse. Let the flux $\vec{\Phi}_s \equiv \overleftrightarrow{Y} \cdot \vec{\Phi}$ with $Y_{i>N_s,j} = 0$, so $\vec{\Phi}_s$ has N_s components.

Only the first N_o of the functions $f_i(\theta,\varphi)$ can be produced efficiently by distant coils. A flux component $(\vec{\Phi}_s)_i$ of $\vec{\Phi}_s$ is determined primarily by the N_o functions if $\sum_{j=1}^{N_o} (Y_{ij})^2 > \frac{1}{2}$. (The factor of one half is to a certain extent arbitrary, but one half is a reasonable requirement.) Let N_e be the number of flux components that satisfy $\sum_{j=1}^{N_o} (Y_{ij})^2 > \frac{1}{2}$. These N_o flux components are the only flux components that can be both efficiently produced by the coils and well represented by the VMEC shape parameters. The other flux components are either inefficiently produced by coils or are poorly represented by the VMEC shape parameters and should be penalized by the target function, Eq. (19), to insure a plasma shape that can be efficiently produced by coils. If $N_e < N_o$, the coils can produce plasma shapes that the shape parameters of the equilibrium code can not accurately represent. In other words, N_e / N_o is the completeness of the shape parameterization in VMEC.

IV. Optimal Coils for Flexibility

The utility of a stellarator experiment is largely determined by the ability of its coils to support a large number of plasma equilibria. Different plasma equilibria mean different profiles of plasma pressure and current as well as more fundamental changes in the configuration. Assume that a large group of i_{eq} equilibria is given which one would like to

support by varying the currents in a set of coils. Each member of this group of equilibria can be made consistent with distant coils using the technique described in Section (III). In this section a method will be given for obtaining a fixed primary coil set plus the smallest number of control coils required support the i_{eq} equilibria. The currents in the control coils have different values for each of the i_{eq} equilibria. The fixed current in the primary coil set is chosen to minimize the current required in each of the control coils.

The first task in the design of a set of coils is the choice of the control surface. The control surface must have a minimum separation from every member of the group of equilibria it is to support. The minimally separated surface is a reasonable starting point for the design of a set of coils. The choice of control surface determines a set of N_o functions $g_j(\theta,\varphi)$, such as the natural functions, that are associated with magnetic fields that have a slow spatial decay, Section (V). Each of the i_{eq} equilibria should be fit as well as possible, using the technique described in Section (III), by a current potential on the coil surface of the form

$$\kappa_i(\theta,\varphi) = \sum_{j=1}^{N_o} I_{ij} g_j(\theta,\varphi) \tag{20}$$

with $1 \le i \le i_{eq}$ numbering the equilibria. The number of equilibria, i_{eq}, is assumed to be large compared to the number of current distributions, N_o, that are retained. Many equilibria will be obtained by tracking a plasma optimum as parameters in the current and pressure profiles are changed, parameters that give the variation in profiles that an experiment must be capable of accommodating.

The coil set can be separated into primary and control coils. The current potential of the primary coil set is

$$\kappa_p(\theta,\varphi) = \sum_{j=1}^{N_o} I_j^{(p)} g_j(\theta,\varphi) \tag{21}$$

with the currents defining the primary set given by either $I_j^{(p)} = \frac{1}{i_{eq}} \sum_{i=1}^{i_{eq}} I_{ij}$ or $I_j^{(p)} = \left\{ (I_{ij})_{\max} + (I_{ij})_{\min} \right\} / 2$ with "max" or "min" meaning the maximum or minimum within the group of i_{eq} equilibria. The second choice for $I_{ij}^{(p)}$ minimizes the control coil currents required to support the full group of i_{eq} equilibria. The first choice is affected less by extreme equilibria in the group. A natural form for the primary coils is a set of modular coils, which means toroidal field coils that are wound with helical bends so they produce not only the net poloidal current G (or equivalently the toroidal flux) but also the current potential $\kappa_p(\theta,\varphi)$. However, many choices for the primary coils are possible. For example, many stellarators use helical coils.

The control coils are designed by writing the current potential that is required for each of the equilibria as

$$\kappa_i(\theta,\varphi) = \kappa_p(\theta,\varphi) + \sum_{j=1}^{N_o} (I_{ij} - I_j^{(p)}) g_j(\theta,\varphi). \tag{22}$$

The control coils must carry currents that produce the important elements of the matrix

$$(\delta \overleftrightarrow{I})_{ij} \equiv I_{ij} - I_j^{(p)}. \tag{23}$$

The important elements of the matrix $\delta \overleftrightarrow{I}$ can be determined using SVD techniques. Write $\delta \overleftrightarrow{I} = \overleftrightarrow{Y}^T \cdot \overleftrightarrow{i} \cdot \overleftrightarrow{Z}$ with $\overleftrightarrow{i}$ a diagonal and $\overleftrightarrow{Y}$ and $\overleftrightarrow{Z}$ orthogonal matrices. The important current distributions are the components of $\vec{g}^{(c)}(\theta,\varphi) = \overleftrightarrow{Z} \cdot \vec{g}(\theta,\varphi)$ associated with large diagonal elements of the matrix $\overleftrightarrow{i}$. Linear algebra insures that the number of diagonal elements of $\overleftrightarrow{i}$ that are sufficiently large to be of importance, J_c, is less than or equal to N_o. The current that must flow in the control coils to produce one of these equilibria is $I_{ij}^{(c)} \equiv (\overleftrightarrow{Y}^T \cdot \overleftrightarrow{i})_{ij}$. The current potential that gives each of the i_{eq} equilibria is then

$$\kappa_i(\theta,\varphi) = \kappa_p(\theta,\varphi) + \sum_{j=1}^{J_c} I_{ij}^{(c)} g_j^{(c)}(\theta,\varphi). \tag{24}$$

The number of independent control coils J_c is determined by the variety of plasma equilibria that one wishes to support. The number J_c cannot be extremely large. J_c cannot be larger than N_o, but practical considerations make J_c even smaller. The number J_c can not be made extremely small if attractive plasma equilibria are to be supported that have the variation in the plasma pressure and current profiles that is to be expected during plasma start-up and in normal plasma operations. An arbitrarily varied group of equilibria can not be supported by a single set of coils. Iteration is required to determine that most varied group of equilibria that can be supported by a practical set of coils.

A complete specification of the current potential requires more than a specification of the primary and the control currents, which are given in Equation (24). A complete specification requires a statement of tolerances, the accuracy with which the various current potential distributions, the g_i, must be produced. These tolerances are given by the acceptable degradations in the target function $T(\vec{s})$ of the physics optimization. The degradations can be found using quality matrix techniques[6] or the related control matrix techniques that have been developed by Harry Mynick and Neil Pomphrey.[12] For high order functions the tolerance is quite loose, approximately proportional to $(b/a)^{\sqrt{N}}$ with *b/a* the ratio of the coil to plasma radius and N the index of g_N. The natural filtering that Laplace's equation has on high order modes leads to this loose tolerance and implies distant coils can be far simpler than nearby coils.

For general coil design work, it may be easier to have the requirements given in terms of the normal magnetic field on the control surface. Equations (9) and (15) imply the magnetic field in the region enclosed by the control surface is uniquely specified by the poloidal current G outside of the control surface and either the normal magnetic field or the current potential on the control surface. That is, the current potential on the control surface and the normal magnetic field are interchangeable. James Bialek has written a code which calculates the relation between the equivalent current and fields. The requirements on the coil system can be specified by

$$\left(\vec{B}\cdot\hat{n}\right)_{control} = B_p(\theta,\varphi) + \sum_{j=1}^{J_c} B_j^{(c)}(\theta,\varphi) f_j^{(c)} + \sum_{j=1}^{\infty} \delta_j f_j(\theta,\varphi). \tag{25}$$

The principal and control field distributions are determined in an obvious way from Equation (24). The tolerances imply an error $\pm\delta_j$ is tolerable in the normal field distribution that has the dependence f_j . The functions f_j can be the natural functions that are discussed in Section (V).

V. Natural Functions on a Surface

The prescription of a surface, $\vec{x}(\theta,\varphi)$, determines a natural set of functions $f_i(\theta,\varphi)$ on that surface, which are solutions to a surface Helmholtz equation, $\nabla_s^2 f_i = -k_i^2 f_i$. This complete, othonormal set of functions is ordered by the wavenumber k_i. When k_i is large compared to the mean curvature at each point on the surface, the solutions to the full Laplacian that have a natural function as a boundary condition grow or decay exponentially away from the surface, $\exp[\pm k_i(r-b)]$.

The natural functions are useful for defining the number of parameters that are available with which to optimize stellarators. The wavenumber of the Nth natural function goes as $k_N \to \sqrt{N}/r_s$ for $N \to \infty$ with r_s a constant. (For a large aspect ratio symmetric torus $(2\pi)^2 r_s^2$ is the surface area.) The number of axisymmetric natural functions is much smaller scaling as $k_N \to N/r_s$ with r_s roughly the minor radius.

The natural functions have an arbitrary weight function $w(\theta,\varphi)$ in their orthonormality relation

$$\oint f_i f_j w da = \delta_{ij} . \tag{26}$$

This weight function has an important interpretation if one assumes the control surface is a thin resistive shell with resistivity η and thickness δ_s. The current potential in the surface can be expanded in terms of the natural functions, $\kappa = \sum I_i f_i(\theta,\varphi)$. If one chooses the weighting $w \propto \eta/\delta_s$, then the Ohmic power dissipated in the surface is $\sum R_i I_i^2$ with the resistance $R_i = \left(\oint(\eta/\delta_s)da\right)k_i^2$. That is, the natural functions are the eigenmodes of the

resistance operator, and the methods of redistributing current to allow better access or lower current density that were discussed in Reference (6) can be implemented using *w*.

Laplace's equation, $\nabla^2 F = \vec{\nabla}\cdot(\vec{\nabla}F)$, can be written as the sum of a surface and a normal part by writing the gradient of the potential as $\vec{\nabla}F = \hat{n}(\hat{n}\cdot\vec{\nabla}F) - \hat{n}\times(\hat{n}\times\vec{\nabla}F)$. The explicit form for the sum is

$$\nabla^2 F = \vec{\nabla}\cdot\left\{\hat{n}(\hat{n}\cdot\vec{\nabla}F)\right\} - \vec{\nabla}\cdot\left\{(\hat{n}\times(n\times\vec{\nabla}F)\right\}. \tag{27}$$

The part of the Laplacian that involves only derivatives within the surface, which is called the surface Laplacian,

$$\nabla_s^2 f \equiv -\vec{\nabla}\cdot\left\{\hat{n}\times(\hat{n}\times\vec{\nabla}f)\right\}, \tag{28}$$

has the natural functions $f_i(\theta,\varphi)$ as its eigenfunctions. In other words, the $f_i(\theta,\varphi)$ obey the in-surface Helmholtz equation

$$\nabla_s^2 f_i = -k_i^2 f_i, \tag{29}$$

which has the k_i^2 as its eigenvalues. The two periodicities of the torus determine all of the boundary conditions.

To proceed further, a choice for the radial coordinate needs to be made. A simple choice for the radial coordinate is

$$\frac{\partial\vec{x}}{\partial r} = w(\vec{x})\hat{n} \tag{30}$$

with $w(\vec{x})$ a weight function that is assumed to be greater than zero. This choice implies $w\hat{n}\cdot\vec{\nabla}F = \partial F/\partial r$. The area element of the surface $\vec{x}(r=b,\theta,\varphi)$ is $d\vec{a} = \vec{\alpha}d\theta d\varphi$ with

$$\vec{\alpha} \equiv \frac{\partial \vec{x}}{\partial \theta} \times \frac{\partial \vec{x}}{\partial \varphi}, \tag{31}$$

the normal to the surface is $\hat{n} = \vec{\alpha} / \alpha$ with $\alpha = |\vec{\alpha}|$, and the coordinate Jacobian is

$$\Im \equiv \frac{\partial \vec{x}}{\partial r} \cdot \left(\frac{\partial \vec{x}}{\partial \theta} \times \frac{\partial \vec{x}}{\partial \varphi} \right) = w\alpha . \tag{32}$$

A different, but standard, form for the area element is $\vec{\alpha} = \Im \vec{\nabla} r$. The consistency of the two forms implies the surface normal $\hat{n} = w\vec{\nabla} r$. The radial part of the Laplacian is

$$\nabla_n^2 F \equiv \vec{\nabla} \cdot \left\{ \hat{n} \left(\hat{n} \cdot \vec{\nabla} F \right) \right\} = \frac{1}{w} \frac{\partial}{\partial r} \left(\frac{1}{w} \frac{\partial F}{\partial r} \right) - \frac{2\mathrm{K}_m}{w} \frac{dF}{dr}. \tag{33}$$

The mean curvature, $\mathrm{K}_m(\theta, \varphi)$, which is the average of the two principal curvatures of the surface, is shown in the Appendix to satisfy

$$\vec{\nabla} \cdot \hat{n} = -2\mathrm{K}_m . \tag{34}$$

Since the natural functions f_i form a complete set, Equation (46), any solution to Laplace's equation can be written as an expansion, $F = \sum u_i(r) f_i(\theta, \varphi)$. The $u_i(r)$ obey coupled ordinary differential equations. For simplicity, assume the weight function is a constant, $w = w_o$, then

$$\frac{d^2 u_i}{dr^2} - k_i^2 u_i = 2 \sum_j \mathrm{K}_{ij} \frac{du_j}{dr} \tag{35}$$

with $\mathrm{K}_{ij} \equiv \oint \mathrm{K}_m f_i f_j w_o da$. The two solutions to Equation (35) are simple when $k_i >> |\mathrm{K}_m|$,

$$u_i(r) \approx e^{\pm k_i (r-b)} \tag{36}$$

in the vicinity of the surface $r=b$. Equation (36) makes the intuitively obvious point that only natural functions that have a long wavelength within the surface r=b are associated with solutions to Laplace's equation that have a slow variation off the surface. The use of the natural functions to define the magnetic fields that have the slowest spatial decay requires that the eigenfunction with the smallest k_i that is neglected have a wavenumber large enough to be approximated by Equation (36). The mixing among distributions that are associated with different spatial decays is irrelevant if all of the distributions that are being intermixed are among the N_o that are in the optimization space.

Using arbitrary coordinates within the surface, the surface Lagrangian is

$$\nabla_s^2 f = \frac{1}{\Im}\left\{\frac{\partial}{\partial\theta}\Big(\Im(\hat{n}\times\vec{\nabla}\theta)\cdot(\hat{n}\times\vec{\nabla}f)\Big)+\frac{\partial}{\partial\varphi}\Big(\Im(\hat{n}\times\vec{\nabla}\varphi)\cdot(\hat{n}\times\vec{\nabla}f)\Big)\right\}. \tag{37}$$

One can use the dual relations of general coordinates to show

$$\hat{n}\times\vec{\nabla}f_i = \frac{1}{\alpha}\frac{\partial\vec{x}}{\partial\varphi}\frac{\partial f_i}{\partial\theta}-\frac{1}{\alpha}\frac{\partial\vec{x}}{\partial\theta}\frac{\partial f_i}{\partial\varphi}. \tag{38}$$

The natural functions can found by a matrix diagonalization. Consider a complete set of functions that are othonormal using the weight w, Equation (26). One such set of functions is

$$\psi_i(\theta,\varphi) = \frac{\sin(m\theta - n\varphi)}{\pi\sqrt{2w\alpha}} \tag{39}$$

and the similar cosine functions. If the stellarators that are being considered have reflection (or stellarator) symmetry, as all major stellarator designs have, one need only consider sinusoidal or cosinusoidal functions, not both. Define a matrix by

$$\Re_{ij} \equiv -\oint w\psi_i\nabla_s^2\psi_j da. \tag{40}$$

Since the Jacobian is $\Im = w\alpha$ and $da = \alpha d\theta d\varphi$, this matrix can be rewritten as a symmetric positive operator on two functions,

$$\Re_{ij} = \Re\left[\psi_i, \psi_j\right] \equiv \oint w(\hat{n} \times \vec{\nabla}\psi_i) \cdot (\hat{n} \times \vec{\nabla}\psi_j) da \,. \tag{41}$$

Since the $\overleftrightarrow{\Re}$ matrix is symmetric, it can be written as $\overleftrightarrow{\Re} = \overleftrightarrow{W}^T \cdot \overleftrightarrow{k}^2 \cdot \overleftrightarrow{W}$ with $\overleftrightarrow{k}^2$ a diagonal and $\overleftrightarrow{W}$ an orthogonal matrix, $\overleftrightarrow{W} \cdot \overleftrightarrow{W}^T = \overleftrightarrow{1}$. The eigenfunctions $f_i(\theta,\varphi)$ are given by $\vec{f}(\theta,\varphi) = \overleftrightarrow{W} \cdot \vec{\psi}(\theta,\varphi)$. These eigenfunctions are orthogonal with weight w since

$$\oint f_i f_j w da = \oint \sum_{i'j'} W_{ii'} \psi_{i'} W_{jj'} \psi_{j'} w da = \sum_{i'j'} W_{ii'} W_{jj'} \delta_{i'j'} = \delta_{ij} \,. \tag{42}$$

The wavenumbers k_i are the square roots of the diagonal elements of $\overleftrightarrow{k}^2$. The eigenfunctions of the surface Helmholtz equation, Equation (29), and their wavenumbers k_i are given by the eigenfunctions $f_i(\theta,\varphi)$ and the eigenvalues of the $\overleftrightarrow{\Re}$ matrix. The use of a very limited orthonormal set of functions $\psi_i(\theta,\varphi)$ defines an approximate solution to Equation (29). Neil Pomphrey has written a code that determines the natural functions using this diagonalization procedure.

Each eigenvalue k_i^2 is an extremum of $\Re[\psi,\psi]$ while holding $\oint \psi^2 w da = 1$ with $\psi(\vec{x})$ otherwise arbitrary. Using a Largange multiplier to enforce the normalization,

$$\delta\left\{\Re[\psi,\psi] + \lambda \oint \psi^2 w da\right\} = -2\oint \delta\psi \left\{\nabla_s^2 \psi - \lambda\psi\right\} w da \,, \tag{43}$$

which proves eigenvalues are extrema. If the directions of principal curvature of the surface are used as coordinate directions, the operator $\Re[\psi,\psi]$ contains terms proportional to $(\partial\psi/\partial\theta)^2$ and $(\partial\psi/\partial\varphi)^2$ but not $(\partial\psi/\partial\theta)(\partial\psi/\partial\varphi)$. By taking the largest or smallest possible values for the various quantities that appear in the integral expression for $\Re[\psi,\psi]$ and using the complete set of functions $\sin(m\theta - n\varphi)$, one finds that constants c_1 to c_4 exist such that

$$c_1 m^2 + c_2 n^2 > k_i^2 > c_3 m^2 + c_4 n^2 \tag{44}$$

with m and n integers. The number of independent functions of the form $\sin(m\theta - n\varphi)$ with $|m| \le m_{\max}$ and $0 \le n \le n_{\max}$ approaches the limit $N = |2m_{\max} n_{\max}|$ for large values of N. This implies the number of eigenvalues that have a value greater than or less than a given eigenvalue obeys

$$k_N \to \frac{\sqrt{N}}{r_s} \tag{45}$$

as N goes to infinity with r_s a distance defined by the spectrum. In a periodic cylinder, the spectrum is $k_i^2 = (m^2/r^2) + (n^2/R^2)$ with $2\pi r$ and $2\pi R$ the two periodicity distances. The radius defined by the spectrum is $r_s = \sqrt{rR}$. If only the toroidally symmetric, $n=0$, eigenfunctions are admitted, as is the case in the design of a tokamak, then far fewer modes are consistent with a small wavenumber, $k_N \propto N$.

Any function $\psi(\vec{x})$ that can be normalized, $\oint \psi^2 w da = 1$, and for which $\Re[\psi,\psi]$ is not infinite can be represented by a series in the eigenfunctions of the operator $\Re[\psi,\psi]$. To prove this let $\psi = \sum_{i=1}^{N} c_i f_i + \Delta_N$ with $c_i \equiv \oint \psi f_i w da$ and $\oint \Delta_N f_i w da = 0$ for $i \le N$. Then, $\Re[\psi,\psi] = \sum_{i=1}^{N} c_i^2 k_i^2 + \Re[\Delta_N, \Delta_N]$, but if the eigenvalues have been properly ordered $\Re[\Delta_N, \Delta_N] \ge k_N^2 \oint \Delta_N^2 w da$, which implies

$$\oint \Delta_N^2 w da \le \frac{\Re[\psi,\psi]}{k_N^2}. \tag{46}$$

As N goes to infinity, the square integral of the error in the approximation to the solution must vanish. In other words, the eigenfunctions of $\Re[\psi,\psi]$ form a complete set of functions.

The matrix $\Re_{ij}$ is proportional to the resistance matrix[6] of a thin shell, $\overleftrightarrow{R}$, in which $w \propto \eta/\delta_s$, the resistivity divided by the shell thickness. More precisely, if one lets the weighting function be

$$w = \frac{\eta / \delta_s}{\oint \eta / \delta_s da}, \tag{47}$$

the eigenfunctions $f_i(\theta,\varphi)$ of $\overleftrightarrow{\Re}$ are dimensionless, and the resistance matrix is

$$\overleftrightarrow{R} = \left(\oint (\eta / \delta_s) da\right) \overleftrightarrow{\Re}. \tag{48}$$

In other words, the Ohmic power dissipated in thin shell by the current potential $\kappa(\theta,\varphi) = \sum I_i f_i(\theta,\varphi)$ is $P = \vec{I}^T \cdot \overleftrightarrow{R} \cdot \vec{I}$.

VI. Discussion

Practical magnetic field coils cannot precisely support a plasma that has a shape prescribed by an optimization of its physics properties. Recently a method was outlined for designing coils that reproduce optimized stellarator configurations within a certain tolerance on the degradation in quality.[6] A different method has been outlined here that constrains the optimization of the stellarator to configurations that require only N_o easily produced distributions of magnetic field. Both methods utilize the target function of the physics optimization $T(\vec{s})$ to define the implications of restricting the number of distributions of field that the coils must produce to N_o.

The proposed method assumes the plasma always has a fixed boundary, which is determined by the values of a set of coefficients, $\vec{s}$. The optimization is carried out within the defined set of coefficients; a well-defined plasma surface always exists. This means the coils that are designed will not always produce a good plasma surface, for the plasma surface was not allowed to break within the formalism. Problems with the quality of the magnetic surfaces can be corrected by the addition of trim coils that produce field components that resonate with the closed magnetic field lines in, and close to, the plasma. To find the strength of the currents required in the trim coils, enforce the plasma boundary that is given by the optimization by placing a fictitious current carrying surface just outside the plasma. The current in this fictitious surface is calculated in the background field of the coils that have been designed. The magnitude of the fictitious current can be slowly

reduced to zero while adjusting the currents in the trim coils to prevent the breakup of the plasma surface or other surfaces in the plasma.

The proposed method of designing stellarator coils couples the design process to the general physics optimization of the stellarator configuration. The method limits that optimization to those configurations that can be produced by practical coils. This restriction, which may at first appear to be a disadvantage, is the fundamental power of the method. As outlined in Section (IV), the method determines an optimal set of primary coils plus the minimal number of control coils that are required to support a broad group of plasma equilibria. The physics and the engineering design of stellarators can be separated within the method. A control surface that lies as close to the plasmas as any coils can produces the separation. The requirements on the coils are specified by giving the poloidal current and the normal field on a control surface that the coils must produce. The normal field has three parts: (1) the primary field, (2) the field distributions that must be variable within a required range, and (3) the allowed tolerances on all distributions of normal field, Equation (25).

A number of persons have asked whether it would not be better to optimize the plasma using a set of coils that have a number of free parameters. A direct optimization, which is a generalization of the method used by Drevlak,[5] would dispense with the need for a control surface. Five issues favor the use of a control surface on which the coil requirements are specified.

1. Completeness: A plasma optimization should consider all magnetic field distributions that can be efficiently produced by coils. All magnetic field distributions are included if the normal field on the control surface is expanded in the natural functions, Section (V). However, it is difficult to insure a complete representation using coils. Approximately thirty field distributions have a slow spatial decay, so the coils must have at least that many free parameters. However, it is difficult to insure that collections of the coil parameters do not form a pseudo-null space, a set of parameters that if varied together produce little change in the field on the plasma.

2. Flexibility: Optimal coils must support not just one, but many, interesting plasma configurations. The number or type of independent coil currents that are required for flexibility is not known a priori. Just having a large number of independent coil currents does not insure useful flexibility. For example, in a tokamak driving the currents

independently in each toroidal field coil adds no useful flexibility. Having a complete, but limited, set of parameters for describing the field distributions is critical to optimizing a device for flexibility, Section (III).

3. Constraint on the plasma/coil separation: As the plasma pressure and current evolve, changes occur in the shape of the plasma. A coil system in which the coils have many free shape parameters must be constrained to remain a minimum distance from all of the required plasma configurations. This constraint defines a surface that the coils cannot penetrate. The constraint surface can be identified with the control surface of this paper.

4. Coil size: An argument for using coils is that a real coil produces some short wavelength fields. Even though these fields decay between the coils and the plasma, they may have a beneficial effect. If true, the coils must be simulated as full bundles and not by a few filaments. This is demonstrated by letting d be the cross-sectional dimension of the coils. A current potential representation of the coils can be expanded in the natural functions, Section (V). Only terms associated with wavenumbers that satisfy $k_i d \ll \pi$ are insensitive to the shape of the coils. A phase shift $k_i d = \pi$ changes the sign of the driven field. The coils are normally made as large as possible in order to minimize the current density, so $k_i d \approx \pi$ for the largest wavenumbers that are explicitly retained.

5. Plasma sensitivity: A field distribution with a rapid spatial decay can be important to an optimization only if the plasma is sensitive to a small change in the distribution. Sensitivity to a field distribution implies control and careful design are necessary. If the plasma optimization is very sensitive to a missing field distribution, that distribution can be found using moderate values of the parameter c_p, Equation (19), and a decision can be made whether that distribution should be added to the control surface specification.

The control surface can be used in two different ways to determine coils. First, the control surface can be used directly as a surface on which a current potential is defined. The contours of constant current potential can then be used to define the turns of the windings. In reality, several such coil surfaces would be required to allow the independent current potentials required for flexibility. Coil optimization with several coil surfaces was used[8] in Wendelstein 7-X. Optimization methods using several surfaces were also discussed in Reference (6). Second, the control surface can be used as the surface on which the normal field that coils must produce is specified, Equation (25). The control surface offers maximal freedom to the coil designer to find the cheapest and most efficient

coils with reasonable plasma access that are consistent with the physics requirements of the device.

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References:

1. F. Bauer, O. Betancourt, and P. Garabedian, "Magnetohydrodynamic equilibrium and stability of stellarators," (Springer-Verlag, New York, 1984)

2. J. Nührenberg and R. Zille, Phys. Lett. A 129, 113 (1988).

3. G. Grieger, W. Lotz, P. Merkel, J. Nührenberg, et al, Phys. Fluids B 4, 2081 (1992).

4. P. Merkel, Nucl. Fusion 27, 867 (1987).

5. M. Drevlak, Fusion Technol. 33, 106 (1998).

6. A. H. Boozer, Phys. Plasmas 7, 629 (2000).

7. W. Dommaschk, Comput. Phys. Commun. 40, 203 (1986).

8. P. Merkel in A Collection of Papers Presented at the IAEA Technical Committee Meeting, VIII Stellarator Workshop at Kharkov, USSR, 27-31 May 1991 (International Atomic Energy Agency, Vienna, Austria, 1991) p. 507.

9. G. H. Golub and C. F. van Loan, Matrix Computations 3rd Edition (Johns Hopkins University Press, Baltimore 1996) p. 70.

10. J. Hadamard, "Lectures on Cauchy's problem in linear partial differential equations," (Yale University Press, New York, 1923), p. 33.

11. S. P. Hirshman, W. I. van Rij, and P. Merkel, Comput. Phys. Commun. 43, 143 (1986).

12. H. Mynick and N. Pomphrey, "Control matrix approach to stellarator design and control," submitted to Physics of Plasmas, Feb. 2000.

13. E. W. Weisstein, "CRC Concise Encyclopedia on Mathematics," (Chapman and Hill/CRC, Boca Raton, 1999). See entry "Mean Curvature" on p. 1152.

Appendix: The Divergence of the Surface Normal

The divergence of the normal can be calculated using the formula from the theory of general coordinates for the divergence,

$$\vec{\nabla}\cdot\hat{n}=\frac{1}{\Im}\left\{\frac{\partial(\Im\hat{n}\cdot\vec{\nabla}r)}{\partial r}+\frac{\partial(\Im\hat{n}\cdot\vec{\nabla}\theta)}{\partial\theta}+\frac{\partial(\Im\hat{n}\cdot\vec{\nabla}\varphi)}{\partial\varphi}\right\} \tag{A1}$$

Using the radial coordinate defined by Equation (30), the coordinate Jacobian, which is given in Equation (32), and the orthogonality relations $\hat{n}\cdot\vec{\nabla}\theta=\hat{n}\cdot\vec{\nabla}\varphi=0$ that follow from Equation (30), one finds

$$\vec{\nabla}\cdot\hat{n}=\frac{1}{w\alpha}\frac{\partial\alpha}{\partial r}. \tag{A2}$$

The derivative

$$\tfrac{1}{2}\frac{\partial\alpha^2}{\partial r}=\vec{\alpha}\cdot\left(\frac{\partial^2\vec{x}}{\partial r\partial\theta}\times\frac{\partial\vec{x}}{\partial\varphi}+\frac{\partial\vec{x}}{\partial\theta}\times\frac{\partial^2\vec{x}}{\partial r\partial\varphi}\right). \tag{A3}$$

Now

$$\frac{\partial}{\partial\theta}\frac{\partial\vec{x}}{\partial r}=\hat{n}\alpha\frac{\partial}{\partial\theta}\left(\frac{w}{\alpha}\right)+\frac{w}{\alpha}\left(\frac{\partial^2\vec{x}}{\partial\theta^2}\times\frac{\partial\vec{x}}{\partial\varphi}+\frac{\partial\vec{x}}{\partial\theta}\times\frac{\partial^2\vec{x}}{\partial\varphi^2}\right), \tag{A4}$$

which implies

$$\vec{\nabla}\cdot\hat{n}=-\frac{1}{\alpha^3}\left(\frac{\partial\vec{x}}{\partial\theta}\times\frac{\partial\vec{x}}{\partial\varphi}\right)\cdot\left\{\frac{\partial^2\vec{x}}{\partial\theta^2}\left(\frac{\partial\vec{x}}{\partial\varphi}\right)^2-2\frac{\partial^2\vec{x}}{\partial\theta\partial\varphi}\left(\frac{\partial\vec{x}}{\partial\theta}\cdot\frac{\partial\vec{x}}{\partial\varphi}\right)+\frac{\partial^2\vec{x}}{\partial\varphi^2}\left(\frac{\partial\vec{x}}{\partial\theta}\right)^2\right\}. \tag{A5}$$

This equation can be rewritten as

$$\vec{\nabla}\cdot\hat{n}=-2\mathrm{K}_m \tag{A6}$$

with K_m the mean curvature of the surface at each point on the surface.[13] A surface in three dimensions has two principal curvatures, and K_m is the average of those two curvatures.