Gyrokinetic calculations of the neoclassical radial electric field in stellarator plasmas

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A novel method to calculate the neoclassical radial electric field in stellarator plasmas is described. The method, which does not have the inconvenience of large statistical fluctuations (noise) of the standard Monte Carlo technique, is based on the variation of the combined parallel and perpendicular pressures on a magnetic surface. Using a three-dimensional gyro-kinetic δf code, the calculation of the radial electric field (E_r) in the National Compact Stellarator Experiment [G. H. Neilson *et al.*, Phys. Plasmas **7**, 1911 (2000)] has been carried out. It is shown that a direct evaluation of E_r based on a direct calculation of the radial particle flux is not tractable due to the considerable noise. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1370363]

I. INTRODUCTION

The lack of toroidal symmetry of stellarators requires a fully three-dimensional description of the plasma. The departure from axi-symmetry in stellarator plasmas leads to enhanced neoclassical losses in the low-collisionality regime. Another related feature of the effect of the non-axisymmetry of the plasma is to strongly modify the drift orbits of the particles (see the reviews by Sadgdeev and Galeev[1] and by Kovrizhnykh[2]).

Over the past few years, there has been a renewed interest in the so-called quasi-axisymmetric (QA) stellarator concept; that is the equilibrium magnetic field strength is approximately symmetric in the magnetic toroidal angle ζ, after transformation to Boozer coordinates.[3] A stellarator experiment based on the QA concepts is currently being designed in the United States;[4] the National Compact Stellarator Experiment (NCSX)[5] is a three-field period, low-aspect-ratio configuration which has good transport and stability properties.[5–7] One important feature of the QA concept is that the plasma can rotate in the direction of quasi-axisymmetry, and it may be possible to exploit and control the formation of transport barriers, as in advanced tokamak plasmas.[8] Since the radial electric field is a major contender[9] in the formation of transport barriers, an accurate calculation of the radial electric field is an important aspect of QA plasmas. The reader who is not familiar with current trends in stellarator design can consult Ref. 10.

In this article we describe a novel method to calculate the neoclassical radial electric field in asymmetric toroidal plasmas. The low-noise method, which exploits the advantages of the δf algorithm,[11–14] is based on the variation of the combined parallel and perpendicular pressures ($P_\parallel$ and $P_\perp$) on a magnetic surface. As an example, we have calculated the radial electric field E_r for the C82 configuration of the NCSX plasma.[5] It is shown that a direct (Monte Carlo) calculation of the radial electric field based on the radial particle flux is not tractable due to the large statistical fluctuations; interestingly, the gyrokinetic calculation using the variation of $P_\parallel$ and $P_\perp$ on a given magnetic surface does not exhibit large statistical deviations, which allows for a determination of E_r.

The article is organized as follows; in Sec. II, we describe the method used to calculate the radial particle flux. The numerical method, the computational details and the results are given in Sec. III. Concluding remarks and future application of the method are presented in Sec. IV.

II. THE METHOD

In this section, we describe the method used to determine the radial electric field based on the local variation of the parallel and perpendicular pressures ($P_\parallel$ and $P_\perp$, respectively). The quantities $P_\parallel$ and $P_\perp$ are evaluated by taking appropriate velocity moments of $\delta f \equiv f - f_0$, where f is the total distribution function whereas f_0 is its equilibrium part (usually f_0 is taken to be a Maxwellian distribution). The perturbed part of the distribution function, δf, evolves due to the combined effects of magnetic drifts and spatial inhomogeneity. Another subtle point regarding the numerical calculation is that the velocity moments for $P_\parallel$ and $P_\perp$ are carried out in small annulus (i.e., finite *volume*) around a magnetic surface of reference; this point is discussed in more detail in Sec. III.

In stellarator geometry, it is convenient, both for analytical and computational purposes, to use magnetic coordinates. The confining magnetic field **B** is written in Boozer coordinates[3] as

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$$\mathbf{B}=\iota(\psi)\nabla\zeta\times\nabla\psi+\nabla\psi\times\nabla\theta,$$
$$\mathbf{B}=g(\psi)\nabla\zeta+I(\psi)\nabla\theta+\beta_\star\nabla\psi, \tag{1}$$

where θ and ζ are the poloidal and toroidal angles, respectively; ψ is proportional to the enclosed toroidal flux; ι is the rotational transform; and $g(\psi)$ and $I(\psi)$ are, within a multiplicative constant, the poloidal and toroidal currents, respectively. In these coordinates the Jacobian of the transformation, $\mathcal{J}\equiv[\nabla\psi\cdot(\nabla\theta\times\nabla\zeta)]^{-1}$, satisfies $\mathcal{J}B^2=g(\psi)+\iota(\psi)I(\psi)\equiv F(\psi)$. Consider the momentum of the balance equation for the ions

$$\rho\frac{d\mathbf{V}}{dt}=-\nabla\cdot\mathbf{P}+en\left(\mathbf{E}+\frac{\mathbf{V}\times\mathbf{B}}{c}\right)+\mathbf{F}+\mathbf{R}, \tag{2}$$

where $\mathbf{V}$ is the fluid velocity, ρ is the mass density, $\mathbf{R}$ is the force due to collisions, $\mathbf{F}$ represents the (possible) applied force, and $\mathbf{P}=P_\parallel\hat{\mathbf{b}}\hat{\mathbf{b}}+(\mathbf{I}-\hat{\mathbf{b}}\hat{\mathbf{b}})P_\perp$ is the pressure tensor; here $\hat{\mathbf{b}}=\mathbf{B}/B$ is the unit vector along $\mathbf{B}$, $P_\parallel$ and $P_\perp$ are, respectively, the parallel and perpendicular pressures. We note that Eq. (2) can be obtained by taking the first-order velocity moment of the kinetic equation in the small gyro-radius limit. Taking the scalar product of $\mathbf{e}_\zeta\equiv\partial\mathbf{r}/\partial\zeta$ (where $\mathbf{r}$ is the position vector) with Eq. (2), and operating with $\langle\ldots\rangle=\int\int\ldots\mathcal{J}(\psi,\theta,\zeta)d\theta d\zeta$, we obtain

$$\frac{\iota(\psi)}{c}\frac{dQ}{dt}-\left\langle\frac{dL_\zeta}{dt}\right\rangle=\left\langle\frac{\partial\hat{P}}{\partial\zeta}\right\rangle-T_\zeta, \tag{3}$$

where $\hat{P}\equiv(P_\parallel+P_\perp)/2$, $T_\zeta=\langle(\mathbf{R}+\mathbf{F})\cdot\mathbf{e}_\zeta\rangle$ is the torque due to applied forces and collisional drag; L_ζ is the toroidal component of the canonical momentum $\mathbf{L}=\rho\mathbf{V}+e\mathbf{A}/c$, where $\mathbf{A}=\psi\nabla\theta-\chi\nabla\zeta$ is the vector potential and $2\pi\chi$ is the poloidal flux. In deriving Eq. (3) we have assumed that the electrostatic potential (yet to be determined) is of the form $\mathbf{\Phi}=\mathbf{\Phi}(\psi)$, and we have neglected the loop voltage ($\partial\chi/\partial t\simeq 0$). Using Eq. (1), the first term on the left-hand side of Eq. (3) can be obtained from

$$\left\langle en\mathbf{e}_\zeta\cdot\left(\frac{\mathbf{V}\times\mathbf{B}}{c}\right)\right\rangle=\frac{e}{c}\int\int n\mathbf{V}\cdot(\mathbf{B}\times\mathbf{e}_\zeta)\mathcal{J}d\theta d\zeta$$
$$=\frac{e}{c}\iota(\psi)\int\mathbf{\Gamma}\cdot d\boldsymbol{\sigma}_n=\frac{\iota(\psi)}{c}\frac{dQ}{dt}.$$

Here Q is the total charge, $\mathbf{\Gamma}=n\mathbf{V}$ is the particle flux, and $d\boldsymbol{\sigma}_n\equiv\mathcal{J}\nabla\psi d\theta d\zeta$ is an area element normal to the magnetic surface $\psi=$const and pointing outwards. The first term on the right-hand side of Eq. (3), which is related to the pressure tensor term in Eq. (2), is derived in the Appendix. The parallel and perpendicular pressures which enter

$$S\equiv\langle\partial\hat{P}/\partial\zeta\rangle$$

are calculated from the velocity moments of δf which, in turn, depends on the particle trajectories. The particle trajectories being affected by the applied electric field $\mathbf{E}=-(d\mathbf{\Phi}/d\psi)\nabla\psi$, we may write $S=S(E_r,t)$. After a few collision times the distribution δf will relax and $S_\infty(E_r)=S(E_r,t\to\infty)$ will provide a measure of the time variation of the charge Q. For one-species simulation, one can then determine the radial electric $E_r^{(0)}$ by solving

$$S_\infty[E_r^{(0)}]=0. \tag{4}$$

Alternatively, one can measure the flux surface-averaged radial particle flux $\langle\Gamma_r\rangle(E_r)$ and determine $E_r^{(0)}$ such that the particle flux vanishes. However, in quasi-axisymmetric toroidal configuration, the large statistical fluctuations in Γ_r can be comparable to, or larger than, the time-averaged signal. As will be shown in the next section, a direct measurement of the radial particle flux is too noisy to be of practical use. Alternatively, a dynamic calculation using the global gyrokinetic toroidal code (GTC),[15] which has been rigorously benchmarked against analytical tokamak neoclassical transport theory,[16] shows that $S(E_r,t)$ reaches a low-noise asymptotic value after a few ion-ion collision times. The general behavior of S is of the form $S(E_r,t)\simeq S_0\times\exp(-Ct/\tau_{ii})+S_\infty[1-\exp(-Ct/\tau_{ii})]$, where τ_{ii} is the ion-ion collision time, $S_0(E_r)=S(E_r,t=0)$, and C is a constant of the order of unity. It is interesting to note that S provides information on the asymmetric part of the particle transport. As it turns out, S strictly vanishes in an axisymmetric configuration (such as an ideal, two-dimensional tokamak plasma). To show this, we use the definition of the flux-surface average and perform an integration by parts:

$$S=-\int\int\hat{P}\frac{\partial\mathcal{J}}{\partial\zeta}d\theta d\zeta=-\left\langle\hat{P}\mathcal{J}^{-1}\frac{\partial\mathcal{J}}{\partial\zeta}\right\rangle.$$

Noting that $\mathcal{J}B^2$ is a flux surface quantity it follows that $\mathcal{J}^{-1}\partial\mathcal{J}/\partial\zeta=-2B^{-1}\partial B/\partial\zeta$ and

$$S=\left\langle\frac{P_\parallel+P_\perp}{B}\frac{\partial B}{\partial\zeta}\right\rangle, \tag{5}$$

showing that $S\equiv 0$ in an axisymmetric configuration. We note that the off-diagonal contributions in the pressure tensor have been neglected (in terms of the smallness parameter ρ_i/R_0). The inclusion of finite Larmor radius (FLR) effects, such as in the paper of Rosenbluth *et al.*,[17] can lead to non-ambipolar transport.

III. NUMERICAL METHOD AND RESULTS

In this section, we describe the low-noise numerical method used to evaluate $S(E_r,t)$, which provides a measure of the radial particle flux. The computational domain and the collision operators are also discussed. Finally we present specific numerical results for the NCSX plasma.

It is convenient to write the parallel and perpendicular pressures in terms of their respective Fourier components (on a given magnetic surface); for example, the perpendicular pressure can be written as

$$P_\perp=\Sigma_{m,n}(P_\perp)_{m,n}\exp[i(m\theta+nN_p\zeta)], \tag{6}$$

where N_p is the number of field periods of the configuration and the Fourier coefficients are calculated according to

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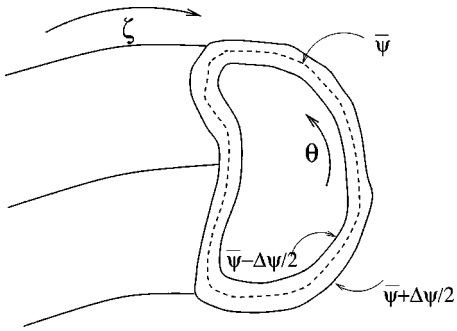

FIG. 1. Computational domain for δf calculations of the radial electric field. The magnetic coordinates are ψ, θ and ζ. The particles are initialized at the toroidal flux surface $\bar{\psi}$. The width of the annulus is such that $\delta\psi/\psi_b \ll 1$, where ψ_b is the toroidal flux at the plasma boundary.

$$(P_\perp)_{m,n} = \frac{\int_0^{2\pi} d\theta \int_0^{2\pi} d\zeta (m v_\perp^2/2)\,\delta f \exp[-i(m\theta + nN_p\zeta)] d^3v}{\int_0^{2\pi} d\theta \int_0^{2\pi} d\zeta}. \tag{7}$$

In practice the particle trajectories are integrated in a volume enclosed between two neighboring toroidal flux surfaces $\bar{\psi} - \Delta\psi/2$ and $\bar{\psi} + \Delta\psi/2$ (Fig. 1). Here $\bar{\psi}$ is a magnetic surface of reference and $\Delta\psi$ is chosen so that $\Delta\psi/\psi_b$, where ψ_b is the toroidal flux at the plasma boundary, is much less than unity. However, due to the combined effects of magnetic drifts and background inhomogeneity, the particles will drift outside the layer $\Delta\psi$, the number of Lagrangian markers within the layer will decrease in time, and the statistics associated with $P_\parallel$ and $P_\perp$ will become poorer. To bypass this difficulty, particles are uniformly loaded in the layer $\Delta\psi$ but the perpendicular and parallel pressures are monitored in the annulus $\delta\psi \ll \Delta\psi$ (Fig. 2). Introducing the radial coordinate $r = \sqrt{\psi/B_0}$ (where as before ψ is the enclosed toroidal flux), one has $\Delta\psi = \psi(r+\Delta r) - \psi(r) \approx 2B_0 r \Delta r$ from which we get $\Delta r \approx \Delta\psi/(2\sqrt{B_0\psi})$. The radial component of the curvature drift velocity is of the order of $V_d \approx v_{th}(\rho_{th}/R)$, where v_{th} is the thermal velocity, ρ_{th} is the associated thermal gyroradius and R is the major radius; here we have used $R_c \equiv |\nabla B/B|^{-1} \sim R$ as the typical scale length of the magnetic field inhomogeneity. Therefore the typical drift time within the layer $\Delta\psi$ is $\tau_d \approx \Delta r/V_d$, which is chosen so that $\tau_d \gg \tau_r$, where τ_r is the relaxation time (typically a few ion-ion collision time). Alternatively, one can assume periodicity in the radial direction, that is the particle that escapes the radial domain $\delta\psi$ can be put back in the same domain; in this case, the number of Lagrangian markers remains constant. Different schemes can be used and compared to randomize the position (ψ, θ, ζ) and the pitch of the particle. However, as in a real experiment, one is interested in the *global* radial profile of the electric field; one important aspect of the method previously described is that it can easily be extended to the entire plasma volume, without additional assumptions regarding the boundary conditions.

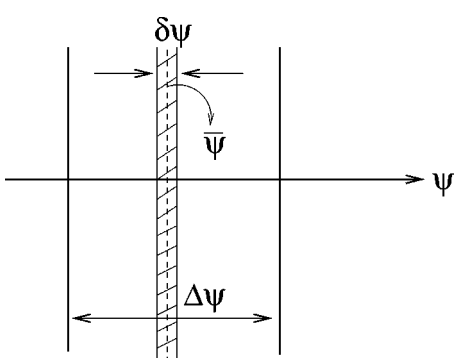

FIG. 2. The Lagrangian δf markers are uniformly distributed in the layer of width $\delta\Psi$; the radial width is chosen such that the typical drift time of an ion in the layer is larger than the relaxation time of the perturbed parallel and perpendicular calculated in the annulus $\delta\psi$ centered at $\bar{\psi}$.

The guiding center motion and the collisions will spread the particles toward equal density in pitch and over the magnetic surface; therefore it is convenient to make the replacement (again using the particular form of the Jacobian in Boozer coordinates and the definition of the volume element in magnetic coordinates, $d^3x = \mathcal{J} d\psi d\theta d\zeta$)

$$\int\int d\theta\, d\zeta \Rightarrow \int \mathcal{J}^{-1} (\delta\psi)^{-1} d^3x \Rightarrow [F(\bar{\psi})\delta\psi]^{-1} \int B^2 d^3x.$$

Therefore one can calculate the Fourier coefficients for the perpendicular pressure according to

$$(P_\perp)_{m,n} = \int \frac{\int d^3x (m v_\perp^2/2)\,\delta f B^2 \exp[-i(m\theta + nN_p\zeta)]}{\int d^3x B^2} d^3v. \tag{8}$$

The same method was used to evaluate the parallel pressure on the magnetic surface $\bar{\psi}$.

As is well known in neoclassical theory, the momentum and energy conservation properties of the collision operator are important for accurate calculation of quantities such as the radial particle flux. The gyrophased collision operator for like-species collisions can be written as[18]

$$C(\delta f) = \frac{\partial}{\partial v_\parallel}(\nu_\parallel \delta f) + \frac{\partial}{\partial v_\perp^2}(\nu_\perp \delta f) + \frac{\partial^2}{\partial v_\parallel v_\perp^2}(\nu_\perp^\parallel \delta f) + \frac{\partial^2}{\partial v_\parallel^2}(\nu_\parallel^\parallel \delta f) + \frac{\partial^2}{(\partial v_\perp^2)^2}(\nu_\perp^\perp \delta f), \tag{9}$$

where

$$\begin{aligned} \nu_\parallel &= \nu_0 v_\parallel F, \\ \nu_\perp &= \nu_0[v_\perp^2(2F - H - G) - 2v_\parallel^2 G], \\ \nu_\perp^\parallel &= 2\nu_0 v_\perp^2 v_\parallel (H - G), \\ \nu_\parallel^\parallel &= \nu_0(v_\parallel^2 H/2 + v_\perp^2 G/2), \\ \nu_\perp^\perp &= 2\nu_0 v_\perp^2 (v_\perp^2 H + v_\parallel^2 G). \end{aligned} \tag{10}$$

In Eq. (10), $\nu_0 = 4\pi n_\beta q_\alpha^2 q_\beta^2 \ln\Lambda_{\alpha\beta}/(m_\alpha^2 v^3)$ is the basic frequency for collisions of test particles α with background particle β. F, G, and H are dimensionless functions that can be written in terms of the Maxwell integral $\Theta(x) = 2\pi^{-1/2}\int_0^x y^{1/2}\exp(-y)dy$, where $x = v^2/v_{th\beta}^2$ and $v_{th\beta}$ is the thermal velocity of the background particles. The functions

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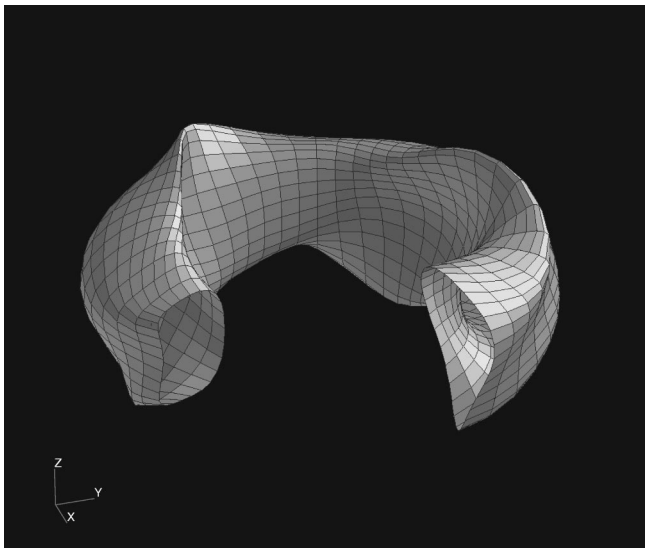

FIG. 3. Magnetic surface of the three-field period National Compact Stellarator Experiment (NCSX). Although the shape of the magnetic surface is largely different for that of a comparable tokamak, the equilibrium B field is approximately symmetric in ζ (magnetic toroidal angle) after transformation to Boozer coordinates.

F, G, and H are $F(x)=(1+m_\alpha/m_\beta)\Theta(x)$, $G(x)=\Theta(x)[1-1/(2x)]+d\Theta/dx$ and $H(x)=\Theta(x)/x$. The test-particle drag and diffusion can be implemented by utilizing a Monte Carlo method due to Xu and Rosenbluth.[18] The particle weights are modified such that the collision operator annihilates a shifted Maxwellian.[16] For ion-electron collisions a Lorentz collision operator is used, and its Monte Carlo implementation has been discussed elsewhere.[19]

For the simulations presented in this article, the trajectories of a set of 2×10^5 Lagrangian markers have been integrated[20–22] with a time step $\Delta t/\tau_{ii}=4\times10^{-4}$. Collisional effects are calculated every ten time steps. The confining B field and the shape of the magnetic surfaces (R,Z,ϕ) have been specified in terms of Fourier series (Fig. 3); a set of 30 Fourier harmonics have been retained in the calculations. Other parameters are the on-axis magnetic field $B_0=1.26\times10^4$ G, central ion temperature $T_i(0)=2.76$ keV, central electron temperature $T_e(0)=2.14$ keV and central plasma density $n_0=6.73\times10^{13}$ cm^{-3} (these parameters are the typical design parameters for NCSX). The magnetic surface of reference is located at $\overline{\psi}/\psi_b=0.7$. At each time step, the local (i.e., within $\delta\psi$; see Fig. 2) perpendicular and parallel pressures from each processor element (PE) are collected onto a single PE (say PE=0); the Fourier coefficients for $P_\parallel$ and $P_\perp$ are then calculated according to Eq. (7). All numerical parallel computations reported here have carried out with 16 PEs.

The radial particle flux obtained from a direct measurement of the radial particle flux (arbitrary units) is shown as the jagged curve in Fig. 4 (in this case the applied radial electric field is 0 and the background distribution function f_0 has been loaded as a Maxwellian with $\langle V_\parallel\rangle=0$). It is noted that the noise is comparable to the signal; to determine the relation $\Gamma_r=\Gamma_r(E_r)$ from a direct measurement is not accurate. As a comparison, the thick line in Fig. 4 represents the

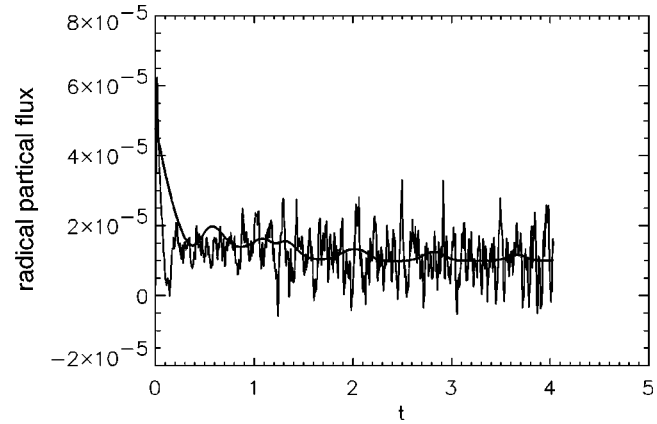

FIG. 4. Radial particle fluxes (arbitrary units) at the magnetic surface $\overline{\psi}/\psi_b=0.7$ as obtained from a direct measurement (thin, broken line) and from the fluid moment approach (thick line). The perpendicular and parallel pressures relax on a time scale of the order of a few ion-ion collision times. The applied radial electric field is zero and the time has been normalized to the ion-ion collision time τ_{ii}.

radial particle flux calculated from the velocity moments $P_\parallel$ and $P_\perp$.

The electron current density (calculated for large t) as a function of the normalized radial electric field $E_r=-\overline{a}d/dr(e\Phi/T_i(0))$ is shown in Fig. 5. The electron current density displays an almost linear dependence on E_r. The ion current density, shown in Fig. 6, shows, however, a strong dependence on the E_r parameter. The largest ion flux is obtained for $E_r\simeq-0.2$. We note that the ion flux is typically two orders of magnitude larger than the electron flux for small electric field (note the scale difference between Figs. 5 and 6). By inverting the relation $\Gamma_i[E_r^{(0)}]=\Gamma_e[E_r^{(0)}]$, we obtained $E_r^{(0)}\simeq-0.87$ statvolt/cm, that is $E_r^{(0)}\simeq-26.2$ kV/m (which corresponds to the stable root). For illustrative pur-

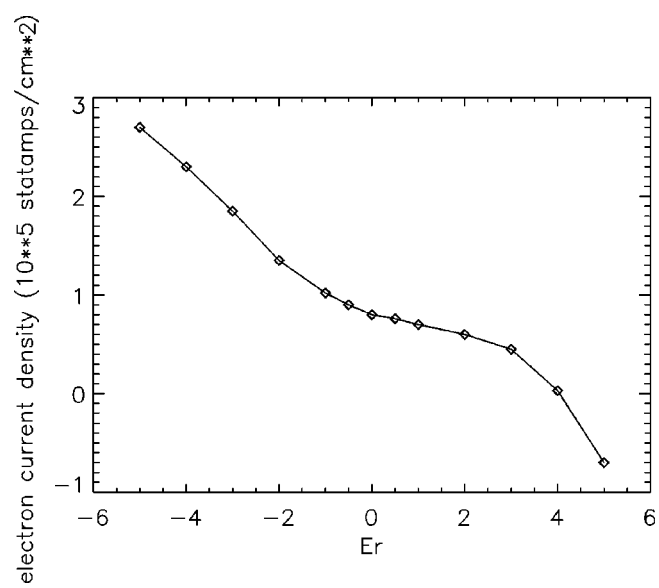

FIG. 5. Electron current density as a function of the normalized radial electric field $E_r=-\overline{a}d/dr(e\Phi/T_i(0))$, where $\overline{a}$ is the average minor radius of the last closed magnetic surface, and $T_i(0)$ is the ion temperature at the magnetic axis.

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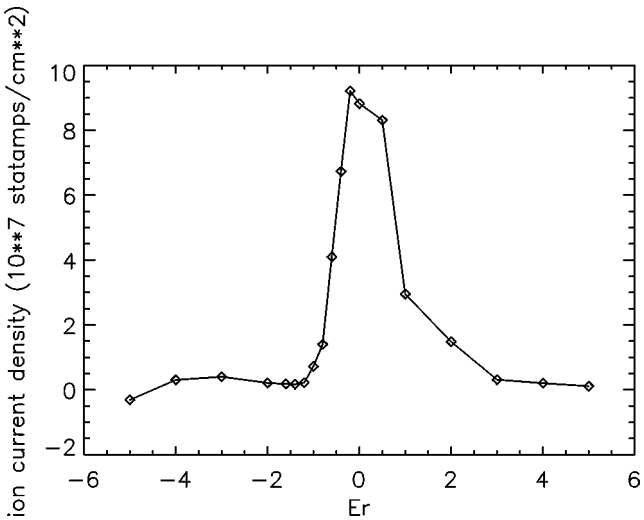

FIG. 6. Ion current density as a function of the normalized radial electric field (plasma parameters are the same as in Fig. 5).

poses, the calculations have been carried out for a single magnetic surface (more precisely, for a single annulus), but a global calculation would lead the variation of E_r with the radial coordinate.

IV. CONCLUDING REMARKS

In this article, we have presented a method to calculate the radial electric field in stellarator plasmas; our method is particularly useful for toroidal plasmas which depart weakly from axisymmetry (e.g., the ''quasi-asymmetric concept''). It is has been shown that a direct measurement of the radial particle flux is very noisy, and not of much practical use for a direct calculation of E_r. The moment approach, however, shows a relatively smooth behavior and reaches an asymptotic value after a few ion-ion collision times.

The method presented in this article can be improved by including the off-diagonal terms (which are $\sim \rho_i/R_0$ times smaller than the diagonal terms) in the pressure tensor $\boldsymbol{\pi}$. However, since the dominant symmetry breaking term in the B_{mn} spectrum is much larger than the smallness parameter ρ_i/R_0, one can, in first approximation, neglect the off-diagonal terms in the pressure tensor $\boldsymbol{\pi}$.

An extension of the method presented in this article can be used to determine the damping rates of toroidal and poloidal flows in stellarator plasmas; this is left for future work.

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APPENDIX: DIVERGENCE OF THE PRESSURE TENSOR

The pressure tensor, $\mathbf{P}$, can be written as

$$\mathbf{P}=P_{\parallel}\hat{\mathbf{b}}\hat{\mathbf{b}}+P_{\perp}(\mathbf{I}-\hat{\mathbf{b}}\hat{\mathbf{b}}), \tag{A1}$$

where $\hat{\mathbf{b}}=\mathbf{B}/B$ is the unit vector along $\mathbf{B}$, $\mathbf{I}$ is the unit dyadic, and $P_{\parallel}$ and $P_{\perp}$ are, respectively, the parallel and perpendicular pressures. We can also write Eq. (A1) as $\mathbf{P}=\tilde{P}\mathbf{BB}+P_{\perp}\mathbf{I}$, where $\tilde{P}\equiv(P_{\parallel}-P_{\perp})/B^2$. The divergence of the pressure tensor then reads

$$\nabla\cdot\mathbf{P}=\mathbf{B}(\mathbf{B}\cdot\nabla\tilde{P})+\tilde{P}\nabla\cdot(\mathbf{BB})+\nabla P_{\perp}, \tag{A2}$$

where the second term on the right-hand side can be calculated using the relation $\nabla B^2/2=\mathbf{B}\times(\nabla\times\mathbf{B})+(\mathbf{B}\cdot\nabla)\mathbf{B}$ so that

$$\nabla\cdot\mathbf{BB}=\tfrac{1}{2}\nabla B^2-\mathbf{B}\times(\nabla\times\mathbf{B}). \tag{A3}$$

Using Ampere's law, $4\pi\mathbf{J}=c\nabla\times\mathbf{B}$, and the radial force balance, $\mathbf{J}\times\mathbf{B}=c\nabla P_0$, one gets $\mathbf{B}\times(\nabla\times\mathbf{B})=-4\pi\nabla P_0$ and the divergence of the pressure tensor now reads

$$\nabla\cdot\mathbf{P}=\mathbf{B}(\mathbf{B}\cdot\nabla\tilde{P})+\tilde{P}\left(\tfrac{1}{2}\nabla B^2+4\pi\nabla P_0\right)+\nabla P_{\perp}. \tag{A4}$$

Taking the scalar product of Eq. (A4) with $\mathbf{e}_{\varphi}\equiv\partial\mathbf{r}/\partial\varphi$ where $\mathbf{r}$ is the position vector and $\varphi=\{\theta,\zeta\}$ one gets

$$\mathbf{e}_{\varphi}\cdot(\nabla\cdot\mathbf{P})=B_{\varphi}(\mathbf{B}\cdot\nabla\tilde{P})+\frac{\tilde{P}}{2}\frac{\partial}{\partial\varphi}B^2+\frac{\partial P_{\perp}}{\partial\varphi}, \tag{A5}$$

where we used the fact that the equilibrium pressure is a flux surface quantity, $P_0=P_0(\psi)$. In Boozer coordinates, the product of the Jacobian and the magnetic field strength squared is a flux surface quantity, that is, $\mathcal{J}B^2=F(\psi)$, where $\mathcal{J}=[\nabla\psi\cdot(\nabla\theta\times\nabla\zeta)]^{-1}$ denotes the Jacobian and $F(\psi)$ is a linear combination of the poloidal current, toroidal current and safety factor. We introduce the flux-surface average operator $\langle\bullet\rangle$ as

$$\langle\bullet\rangle\equiv\int\int\mathcal{J}(\bullet)d\theta d\zeta. \tag{A6}$$

It is easy to show that $\langle\bullet\rangle$ annihilates the $\mathbf{B}\cdot\nabla$ operator, that is,

$$\langle\mathbf{B}\cdot\nabla G\rangle=0 \tag{A7}$$

for any function $G=G(\psi,\theta,\zeta)$. Noting that, in Boozer coordinates, B_{θ} and B_{ζ} are flux-surface quantities, the flux-surface average of Eq. (A5) is

$$\langle\mathbf{e}_{\varphi}\cdot(\nabla\cdot\mathbf{P})\rangle=\frac{1}{2}\left\langle\frac{\partial}{\partial\varphi}(P_{\parallel}+P_{\perp})\right\rangle, \tag{A8}$$

where we used Eq. (A7) and

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\begin{aligned}
\left\langle \tilde{P}\frac{\partial B^2}{\partial \varphi}\right\rangle &= \left\langle \frac{P_\parallel - P_\perp}{B^2}\frac{\partial B^2}{\partial \varphi}\right\rangle \\
&= \left\langle B^2(P_\perp - P_\parallel)\frac{\partial}{\partial \varphi}\left(\frac{1}{B^2}\right)\right\rangle \\
&= \left\langle \frac{F(\psi)}{\mathcal{J}}(P_\perp - P_\parallel)\frac{\partial}{\partial \varphi}\left(\frac{1}{B^2}\right)\right\rangle \\
&= F(\psi)\int\int (P_\perp - P_\parallel)\frac{\partial}{\partial \varphi}\left(\frac{1}{B^2}\right) d\theta\, d\zeta \\
&= F(\psi)\int\int \frac{1}{B^2}\frac{\partial}{\partial \varphi}(P_\parallel - P_\perp) d\theta\, d\zeta \\
&= \int\int \mathcal{J}\frac{\partial}{\partial \varphi}(P_\parallel - P_\perp) d\theta\, d\zeta \\
&= \left\langle \frac{\partial}{\partial \varphi}(P_\parallel - P_\perp)\right\rangle. \qquad \text{(A9)}
\end{aligned}
$$

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