Magnetics measurements in NCSX: 
SVD/PCA methods-I

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Introduction

A variety of magnetic diagnostics (MD’s) will be installed in NCSX and used for

- Shape Control (during shots)
- Equilibrium Reconstruction (between shots)

1) Define a method for selecting a “good” set of MD’s (optimizes the invertible information)
2) How much info. is available directly from external MD signals? – more than tokamaks?

This presentation is a partial coverage of topic 2.
**Target Function on Control Surface (CS)**

- A database of VMEC equilibria (many hundreds) is being generated with a wide range of shapes and profiles.
- B-fields from each of the equilibria are calculated on a single “Control Surface” (CS) that lies 1cm outside the envelope of all equilibria. These are distributions, $b_j(q,f)$, $j = 1 \ldots N_{eq}$
- Magnetic signals $d_j(x_d)$ are also calculated for each candidate diagnostic using V3RFUN and V3POST.
- The $b_j(q,f)$ are targets for the $d_j(x_d)$. If the targets are reproduced with adequate precision the MD’s should provide sufficient information for control and equilibrium reconstruction!
The Equilibria in the Database have a wide variety of shapes

A surface is defined (the “Control Surface”, CS) which encloses all plasmas in the database. It lies 1cm outside of the envelope of all equilibria. The $B_\perp$ from each equilibrium is calculated by V3RFUN/V3POST and stored for analysis.
Current Profiles in Database

Profiles with $\langle J.B \rangle(s) > 0$ for all $s$
(last 3 digits of AC id shown)

Profiles with $\langle J.B \rangle(s)$ changing sign at some $s$
Pressure Profiles in Database

Profiles with $P_{edge} = 0$
(last 3 digits of AM id shown)

Profiles with $P_{edge}$ finite

![Graph showing pressure profiles with $P_{edge} = 0$ and $P_{edge}$ finite](image)

![Graph showing pressure profiles with $P_{edge} = 0$ and $P_{edge}$ finite](image)
Equilibrium Database Parameters

- $B_r(T)$ vs $I_p(\text{kA})$
- $i_{\text{edge}}$ vs $i_{\text{axis}}$
- $A$ vs $R_0(\text{cm})$
- $\text{height}_{f=0}(\text{cm})$
- $b(\%)$ vs $I_p(\text{kA})$
- $<a>(\text{cm})$ vs $R_0(\text{cm})$
- $\text{height}_{f=p}(\text{cm})$ vs $R_0(\text{cm})$
- Waist$_{f=p}(\text{cm})$ vs $R_0(\text{cm})$
Data Preparation and Expansion in EOF’s

- For each equilibrium, labelled by index j, calculate $B_\perp$ on a uniform mesh of M points on the CS. (This is $b_j(q,f)$)
- Store the signal as an M-element column vector $x_j$.
- Data from $N_{eq}$ equilibria naturally forms an $M \times N_{eq}$ matrix, $X$.
- Each column of $X$ ($B_\perp$ signal on the CS) has an exact expansion as a linear combination of $\min(M,N_{eq})$ orthogonal patterns (called Empirical Orthogonal Eigenfunctions (EOFs))
- The EOFs are eigenfunctions of the correlation matrix $C = X X^T$.
- The calculation of EOFs is most conveniently done by Singular Value Decomposition of $X$
**Singular Value Decomposition (SVD), Principal Components, and EOF's**

**SVD:**

\[
X_{M \times N_{eq}} = U_{M \times M} W_{M \times N_{eq}} V_{N_{eq} \times N_{eq}}^T
\]

equivalent to

\[
x_j = \sum_{k=1}^M (V_{jk} W_k) u_k
\]

\[
\equiv \sum_{k=1}^M Z_k(j) u_k
\]

j\textsuperscript{th} column of X  

\(\equiv \text{k\textsuperscript{th} column of U: Empirical Orthogonal Functions (EOFs).}\)

\(\text{Score for j\textsuperscript{th} observation on k\textsuperscript{th} Principal Component (PC)}\)

**Note:**

\[
x_j^{(\ell)} = \sum_{k=1}^M Z_k(j) u_k, \text{ minimizes } e = \|x_j - x_j^{(\ell)}\|
\]
Interpretation of Principal Components

- Since \( x_j = \sum_{k=1}^{M} Z_k(j) u_k \),
- orthonormality of the EOFs =>
  \[ x_j \cdot u_k = Z_k(j) \]

LHS is essentially an overlap integral.

For \( X = \text{matrix of } B_\perp \text{ signals, } \) Principal Component ”score”
\( Z_k(j) = \iint dqdf \ B_\perp(j)(q,f) \ u^{(k)}(q,f) \) measures importance of \( k^{th} \) EOF in determining the shape of \( j^{th} \) equilibrium signal on the CS.
Interpretation (Cont)

- SVD on $X = U W V^T$ can be interpreted as a variable transformation $X := Z = U^T X \ (= WV^T)$

$$Z_{1j} = U_{11} X_{1j} + U_{21} X_{2j} + U_{31} X_{3j} + ... \ (= W_{1j} V_{j1})$$
$$Z_{2j} = U_{12} X_{1j} + U_{22} X_{2j} + U_{32} X_{3j} + ... \ (= W_{2j} V_{j2})$$
$$Z_{3j} = U_{13} X_{1j} + U_{23} X_{2j} + U_{33} X_{3j} + ... \ (= W_{3j} V_{j3})$$

... ... ... ... ... etc ... ... ... ... ... ...

- The $U_{ij}$ are weights which measure the contribution of each of the original variables to the variance of the data in the transformed coordinates $\rightarrow$ means for selection/rejection of candidate magnetic diagnostics.
Singular Value Analysis of Vacuum Signal
\((B_{\perp \text{Total}} - B_{\perp \text{Plasma}})\) on Control Surface

- Find 8 significant singular values, corresponding to 8 independent, orthogonal, patterns of \(B_{\perp}\) on the CS.
- The 8 patterns are to be expected because we have 8 equilibrium coil current groups (M1-3, TF, PF3-6).

504 equilibria in database
Correlation EOF’s for Vac. Signal on CS (#s 1, 2)
First 2 dominant patterns of \( B_{\perp}^{\text{plasma}} \) contribute 94.5% of total variance. Next 3 contribute an additional 4.0%. Speculate 2, and possibly up to 5, combined moments of \( p(s) \) and \( <J.B>(s) \) may be measurable.
How many independent pieces of information (PC’s) are available from the Data Matrix?

Several procedures (mainly ad hoc)

- **Broken Stick Rule:** Retain principal components for which
  \[ L(k) = \left( \frac{w_k}{w_1} \right)^2 > \frac{1}{N} \sum_{j=1}^{N} s_j^{-1} \] (expected length of k\textsuperscript{th} longest segment of a stick of unit length broken at random into N segments)

- **Average Based:** Retain PC’s for which \( w_k^2 > 0.7 < w_j^2 \)

- **Cumulative Variance Based:** Retain as many PC’s as necessary to bring the cumulative variance of the retained PC’s up to some desired value, say 95% of the total variance – i.e., find k such that
  \[ \frac{\sum_{j=1}^{k} S_{w_j}^2}{\sum_{j=1}^{N} S_{w_j}^2} > 0.95 \]
Correlation EOF's for Plasma Signal on CS (#s 1, 2)

More structure (shorter wavelength) seen here than for vacuum signal – simply distance attenuation?
Each dot represents a particular equilibrium.

All equilibria in the 07/15/04 database are shown.

As an attempt to discover what properties of the equilibria are responsible for the dominant $B_\perp$ signal patterns, try color coding.
Would like to see a separation of colors along one or other of the axes, thereby associating a particular pattern of $B_\perp$ with a profile parameter variation. A sensible guess for the appropriate profile parameters for color separation are $\ell_i/2$ and $b_{\text{pol}}$. 

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**PC2 vs PC1 with Color Coding for Various Plasma Properties**

Sort by color on J.B profile shape (only profiles with J.B > 0 everywhere)

Sort by color on P profile shape (only profiles with P edge = 0)
Current and Pressure Profiles in Database

Profiles with NO Current Reversal

Profiles with Current Reversal

Pressure Profiles with p(1)=0

Pressure Profiles with p(1) finite
PC2 vs PC1 with Color Coding for Various Plasma Properties

Sort by color on value of $b$

Sort by color on value of $I_p$

$b$ and $I_p$ not individually responsible for the two dominant $B_\perp$ patterns.
Why should we think any of this is possible?

- Because of results obtained from a similar analysis using equilibria from the plasma flexibility studies made in preparation for the CDR. There, we had a much more limited set of equilibria, but with the advantage of systematic variation of profiles and plasma parameters.
As a test case, consider magnetic measurements for 2 groups of equilibria where the current profile is varied at fixed $I_p$ and $b$.

1st group: 6 equilibria where $<J.B>$ is varied in core region (red)

2nd group: 2 equilibria where current is added to the edge (blue)

Can we detect current profile variation from external magnetic measurements?
PCA provides a method for distinguishing equilibria

- The $B_\perp$-matrix on the CS is analysed by Singular Value Decomposition.
  $X = U W V^T$

- According to this decomposition, the columns of $U$ (denote by $u_k$) provide a basis for the expansion of any of the field patterns (columns of $X$) on the CS.

- The $u_k$ are called Empirical Orthogonal Functions (EOF’s) and the coefficients of the $\{u_k\}$ are called Principal Components.

- A linear combination of the first few EOF’s can describe much of the variation in the data.

A 2D scatterplot of the first two PC’s of the $B_\perp$-matrix data corresponding to the 8 equilibria in the J-profile scan distinguishes equilibria where the current profile was varied in the core (red cluster) and equilibria where edge current was added (blue cluster).
Note: The plasma boundary shape variation is very small between these 8 equilibria.

- Therefore if analysis of the $B_q$ signals on the CS can distinguish between these equilibria, it is mainly due to the profile variation, not the consequent shape variation.
The $B_\perp$ pattern change is subtle.
Principal Component Analysis (PCA) can distinguish the equilibria.
Projection onto plane of the leading 2 principal components separates an $I_p$-$b$ equilibrium sequence with constant $<J.B>(s)$ and $p(s)$ profiles.
Difference between $B_\perp$ signals on CS for $a=0.0$ and $a=0.5$