# Three-dimensional codes to design stellarators

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Equilibrium, stability, and transport in stellarators can be studied theoretically by running large computer codes. Some of the codes have excellent resolution, and experimental data have been used to validate the results of numerical calculations. An analysis has been made of recent measurements from the Large Helical Device experiment in Japan to see how they compare with the theory [A. Komori, H. Yamada, O. Kaneko *et al.*, Plasma Phys. Control. Fusion **42**, 1165 (2000)]. Observations of confinement time at low collisionality like that in a reactor agree well with estimates using a quasineutrality algorithm to determine the electric potential. A nonlinear magnetohydrodynamic calculation of stability for various pressure profiles gives predictions of the beta limit that are consistent with the observations. Correlation of the theory with measurements justifies using the codes as a tool to design quasisymmetric configurations for a modular stellarator experiment promising better performance at reactor conditions. 
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## I. COMPUTATIONAL METHODS IN FUSION SCIENCE

### A. Introduction

Mathematical research on equilibrium, stability and transport in stellarators and advanced tokamaks has led to the design of configurations now proposed by the fusion community for a proof of principle stellarator experiment. Large-scale computation makes it possible to deal successfully with physics issues that become very complicated in three-dimensional (3D) geometry. Compact stellarator configurations with simple modular coils and only two or three field periods are being designed that have both good stability and quasiaxial symmetry providing adequate transport at reactor conditions. If the bootstrap current is large a three field period device becomes attractive, but if it is small then a solution with two field periods and twelve modular coils may work better. Other good designs with quasihelical symmetry and four or five periods are available whose properties are less sensitive to the bootstrap current. Satisfactory performance requires that there be a magnetic well in the vacuum field, which is a property lacking in some of the candidates that have been considered for the proof of principle experiment. We investigate stability for these configurations by means of a mountain pass theorem asserting that if two solutions of the problem of magnetohydrodynamic (MHD) equilibrium can be found then there has to exist an unstable solution, too.

Another concern is the problem of transport in plasma physics. Maxwell's equations must be coupled to the partial differential equations of fluid dynamics, and that leads to a singular perturbation involving quasineutrality of the electrons and ions. We model this phenomenon by a Monte Carlo method that provides good agreement with experimental measurements.[1] The computer code to implement the model is validated by applying it to analyze new measurements of confinement time at low collisionality that are now becoming available from the Large Helical Device (LHD) stellarator experiment at the National Institute for Fusion Science (NIFS) in Japan.

A variety of computer codes are used in formulating proposals for proof of principle or concept exploration stellarator experiments, and some of the predictions that have been made do not agree. Consequently it becomes necessary to benchmark the codes and compare the numerical methods that have been employed. Such studies should bring to the attention of the fusion community some of the advantages of the mathematical models implemented in the NSTAB equilibrium code and the TRAN Monte Carlo code that we have developed.[2,3] More specifically, nonlinear stability results from NSTAB computations may be more realistic than local criteria evaluated by less accurate techniques, and they are well correlated with existing experiments.

Our 3D computer codes have been applied to design compact stellarators with modular coils whose physical properties compare favorably with those of tokamaks. Quasiaxial symmetry of the magnetic spectrum restricts the orbits of charged particles and leads to good confinement of hot particles at reactor conditions. Because of a similarity to tokamaks that are prone to disruptions, there is concern in the fusion community about MHD stability of the new configurations. We have addressed this issue by making extensive runs of the NSTAB computer code, including examples that compare favorably with experimental measurements or with other numerical calculations. In the cases we consider the contribution to rotational transform from bootstrap current is at a level where disruptions and free boundary kink modes do not seem to be dangerous.

### B. The stellarator concept

The stellarator concept was developed fifty years ago at the Princeton Plasma Physics Laboratory (PPPL) as a secret project with tentative applications to a neutron source. After declassification the C-Stellarator was built there during the

 

1960s, but the construction was so complicated that the experiment faltered, leading to long term disrepute for the concept in this country. Abroad stellarator research thrived in a formulation based on the stellarator expansion and exact solutions of partial differential equations with two-dimensional symmetry in straight geometry.[4] By 1980 very successful experiments were completed in both Germany and Japan. At that point the analytical theory was superseded by numerical methods, including some of ours, which took advantage of the rapid advances that were being made in computer hardware. This resulted in the design of new configurations optimizing equilibrium, stability and tranport by appropriate choice of the shape of the separatrix in space of three dimensions. The large LHD experiment in Japan is the modern implementation of the more conventional theory, while the Wendelstein 7-X (W7-X) experiment being constructed in Germany by the European fusion community is a product of the computer codes developed by more elaborate mathematical methods. Here we are primarily concerned with a future generation of configurations that are designed to meet the requirements for a fusion reactor by exploiting an approximate two-dimensional symmetry of the magnetic structure. These stellarators are analyzed by running the NSTAB and TRAN computer codes.

It is best to choose toroidal geometry for the magnetic confinement of plasma in a fusion reactor so that the orbits of hot ions and electrons will not escape too rapidly. In tokamaks the poloidal field required for equilibrium is produced by net toroidal current that is destabilizing. Stellarators instead provide for the restoring force of the poloidal field by means of 3D geometry that causes the magnetic lines to spiral around on nested flux surfaces where they acquire rotational transform in a less obvious fashion. The conventional way to generate a stellarator field is by means of helical coils that become interlocked as they rotate around the torus. However, better configurations have been found by shaping the plasma so that the external magnetic field confining it is produced by a system of twisted modular coils not unlike the toroidal field coils of a typical tokamak. Significant mathematical questions arise from the challenge of designing stellarators for practical engineering construction. Much of the reluctance of the plasma physics community to accept this more stable concept stems from concern about technological complications with 3D geometry. A sound theory based on mathematical analysis and scientific computing is required to make a convincing case for new stellarator experiments.

Techniques have been developed to analyze both local and global MHD stability of 3D configurations. A fast computational method involving bifurcated equilibria has been found to study nonlinear stability of modes with ballooning structure. Results of these calculations have been compared with experimental measurements from the Compact Helical System (CHS) stellarator at NIFS and with more recent data from the larger LHD stellarator there.[5] One overall conclusion is that the predictions of local stability theory are too pessimistic. For the LHD even global calculations exhibit an MHD mode that is unstable in a case observed at an average $\beta$ of 2.4%, but the ripple of the bifurcated equilibrium that was obtained is restricted to a very narrow shell surrounding the most dangerous resonant surface. Good confinement in an outer region of high shear produces broad pressure profiles in the experiment that raise the $\beta$ limit significantly above levels suggested in earlier theoretical work.

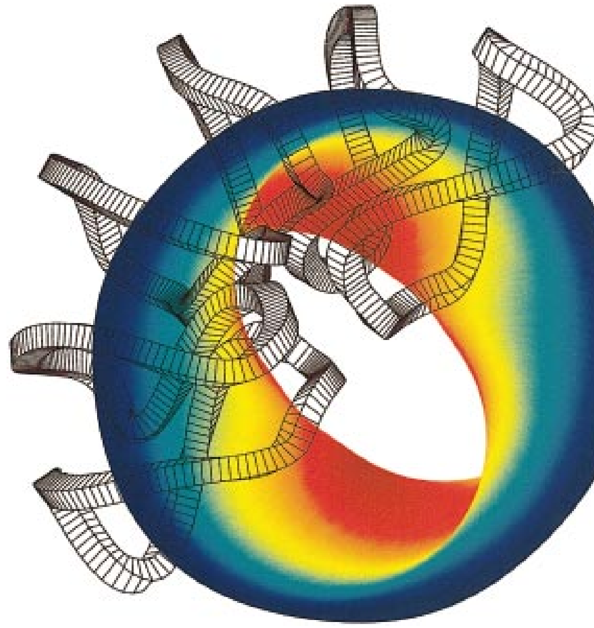

FIG. 1. (Color) Computational model of fusion plasma in a thermonuclear reactor based on the stellarator concept for magnetic confinement of hot ions and electrons. Twelve moderately twisted modular coils, half of which are plotted, produce a magnetic field whose strength has a desirable symmetry displayed by the color map of the plasma surface.

New quasiaxially symmetric (QAS) configurations with two or three field periods and simplified magnetic structure have been found for a proof of principle stellarator experiment. Our mathematical theory, which captures islands by calculating weak solutions of the MHD equilibrium equations, suggests that these are better candidates than those developed by other methods with less resolution. We have collaborated with Shoichi Okamura at NIFS to design a modular stellarator experiment there based on such a configuration (cf. Fig. 1).

The Monte Carlo model of transport implemented in the TRAN code applies well to stellarators. The problem is treated as a random walk among drift surfaces, and details of the geometry are handled computationally by an ordinary differential equation solver. We have found numerical evidence that conservation of momentum is a secondary issue in the calculations and cannot account for quasineutrality, which we take as a requirement that determines the electric field. Resulting perturbations of the electric potential in three dimensions simulate turbulence triggered by the displacement current. Many models of turbulence involve a factor like the eddy viscosity that needs to be determined from additional information. In the TRAN code we compute not only the ion confinement time but also the electron confinement time, which allows us to determine auxiliary factors from quasineutrality. By performing numerical simulations of new experimental measurements from the LHD stellarator at NIFS for low-collision frequencies near those of a reactor we have obtained confirmation of this theory that is more convincing than previous comparisons with tokamak data that required novel 3D perturbations of the magnetic field.

We have applied our Monte Carlo simulation of transport to modular stellarators of the kind discussed above to show that it is possible in such configurations to reach high electron temperatures efficiently by exploiting quasiaxial symmetry of the magnetic spectrum. At very low collision frequencies less than those recently observed in the LHD experiment conventional stellarator scaling laws for the energy confinement time break down because of rapid losses of hot trapped electrons. This means that to raise the peak temperature from the level already achieved in the LHD to values above 5 keV may require much more heating power. No comparable phenomenon should occur in our modular configurations because of two-dimensional quasisymmetry of the magnetic spectrum, so there is a possibility that good transport at low collisionality might produce a peak temperature as high as 10 keV economically.

## II. MAGNETOHYDRODYNAMIC EQUILIBRIUM AND STABILITY

### A. Introduction

Large computer codes have become an accepted tool for the analysis of equilibrium and stability in plasma physics.[6] The NSTAB spectral code is a member of the BETA, BETAS, and VMEC code family which combines a new treatment of the magnetic axis position with improved optimization for high-performance computing. In BETA the finite element method was applied to the MHD variational principle to test for stability by looking for minima in the energy landscape.[7] Weak solutions of equilibrium equations in conservation form were calculated in a mixed Eulerian and Lagrangian coordinate system, but the accuracy was not very good. Meanwhile because of its success with the Navier–Stokes equation, the spectral method had been used in studies of the partial differential equations of resistive MHD. The VMEC code combined these ideas to arrive at a simulation of toroidal confinement of a plasma that has been used extensively to analyze experiments.[8] Faster convergence and better resolution were achieved in the BETAS code, but that did not receive wide acceptance by the fusion community.[9] Further improvements have been made in the NSTAB version, which represents the boundary of the plasma by formulas conveniently related to the magnetic spectrum. We shall describe applications of NSTAB to stellarator and tokamak equilibria with parameters of current experiments.

Nonlinear stability can be investigated with 3D equilibrium codes by either surveying the energy landscape or by looking for bifurcated equilibria that violate uniqueness of the solution. A more quantitative stability criterion is based on the amount of ripple of the flux surfaces that results from an inhomogeneous forcing term in the magnetostatic equations. Incompatible ripple of the surfaces near the fixed boundary of the plasma suggests the attainment of a soft beta limit. For the stellarators we shall be concerned with this estimate of nonlinear instability is more realistic than limits set by local stability criteria.

### B. Nonlinear stability

Equilibrium and stability of tokamaks and stellarators are usually studied by solving the magnetostatic equations

$$\mathbf{J}\times\mathbf{B}=\nabla p, \quad \nabla\cdot\mathbf{B}=0,$$

expressing balance of the Lorentz force against the fluid pressure $p$, where $\mathbf{B}$ is the magnetic field and $\mathbf{J}=\nabla\times\mathbf{B}$ is the current density. We introduce the toroidal flux $s$ such that $\mathbf{B}=\nabla s\times\nabla\theta$ as a radial coordinate, where $\theta$ is a multiple-valued flux function, and we denote by $u$ and $v$ poloidal and toroidal angles on each magnetic surface $s=\text{const}$. In a cylindrical coordinate system $r,\ \vartheta=v/Q$ and $z$ we put

$$r+iz=r_0+iz_0+R(s,u,v)[r_1+iz_1-r_0-iz_0],$$

where the functions $r_1$ and $z_1$ specify an outermost separatrix at $s=1$ with

$$r_1+iz_1=e^{iu}\sum \Delta_{mn}e^{-imu+inv},$$

where $r_0=r_0(v)$ and $z_0=z_0(v)$ describe the location of the magnetic axis, and where $Q$ is the number of field periods. The NSTAB code is based on minimizing the potential energy

$$E=\int\int\int [B^2/2-p(s)]dV,$$

as a functional of $R,\ \theta, r_0$, and $z_0$ subject to appropriate flux constraints.

Euler–Lagrange equations for $R$ and $\theta$ derived from the variational principle are equivalent to the magnetostatic equations, and they can be solved by the spectral method so reliably that it suffices to perform calculations on a single mesh. It follows from the equations that the magnetic field can be expressed in the alternate form

$$\mathbf{B}=\nabla\phi+\zeta\nabla s,$$

using Clebsch potentials $\phi$ and $\zeta$, and both the pressure $p$ and the rotational transform $\iota$ are functions of $s$ alone. When a nested surface hypothesis is imposed on the tori $s=\text{const}$ it is weak solutions of the problem that must be calculated. These are discontinuous solutions of the conservation laws that capture islands and current sheets.

Finite element implementations of the MHD variational principle enable one to assess the stability of toroidal equilibria by examining the energy levels of neighboring configurations. The spectral method, which provides much better resolution in other respects, fails to deliver convincing evidence of stability based on estimates of the energy because of truncation errors associated with determining the magnetic axis. This is true of the NSTAB code in spite of a novel treatment of the magnetic axis which seems to define the shape of nearby flux surfaces quite well. Hence we evaluate nonlinear stability by searching for bifurcated equilibria through an application of the method of steepest descent after perturbations have been introduced that are related to conventional resonant modes. In appropriate coordinates the spectral representation is so accurate that the existence of multiple solutions found this way gives a reliable test for the most important instabilities, for if there are many solutions at least one will be a saddle point of the energy landscape.

An effective procedure is to iterate the calculation of the equilibrium until an apparently stationary value of the energy is reached and then introduce an inhomogeneous term $f$ in the partial differential equation for $R$ that triggers a mode whose stability is to be tested. If perceptible ripple in the magnetic surfaces appears that can be identified with the mode that was triggered, the configuration may be unstable. The ratio of the perturbation of the solution $R$ to the amplitude of the inhomogeneous term $f$ provides a quantitative measure of instability that is helpful in marginal cases.

In stellarator applications some correlation has been observed between limits on average $\beta=2p/B^2$ determined by nonlinear stability and $\beta$ limits based on the Mercier local stability criterion. We therefore calculate the Mercier criterion in the NSTAB code as a sum of the form

$$\Omega=\frac{s^2}{\pi^2\,\iota^2}[\Omega_s+\Omega_w+\Omega_c]>0,$$

where $\Omega_s$ is a term containing the shear $\iota'=\iota'(s)$, $\Omega_w$ is a term measuring the magnetic well $-V''/V'$, and $\Omega_c$ is a term measuring the parallel current $\mathbf{J}\cdot\mathbf{B}/B^2$. Here $V=V(s)$ is the volume inside the torus $s=$const. Convergence studies and comparisons with stellarator experiments show that both the Mercier criterion and a ballooning stability criterion of similar structure are often too pessimistic, and our nonlinear test seems to provide a better prediction of MHD stability. Moreover ambiguity may occur in working with the standard differential equation for ballooning modes because the answer is deceptively unstable for a long path of integration, but is always stable for a very short path, so the result depends on the choice of the length of an arc of the magnetic line being examined.[10]

After renormalization of the flux coordinates $\theta$ and $\phi$ and the toroidal current $I=I(s)$, we represent the magnetic field strength by means of a Fourier series

$$\frac{1}{B^2}=\sum B_{mn}\cos\left[\frac{(m-In)\,\theta+(n-\iota m)\,\phi}{1-\iota I}\right],$$

where the spectrum of coefficients $B_{mn}$ are functions of the radial coordinate $s$. The parallel current is calculated from a truncated and filtered version of the resulting formula

$$\frac{\mathbf{J}\cdot\mathbf{B}}{B^2}=p'\sum\frac{m-In}{n-\iota m}B_{mn}\cos\left[\frac{(m-In)\,\theta+(n-\iota m)\,\phi}{1-\iota I}\right],$$

which involves small denominators. If no filtering is applied, then in 3D equilibria the singularities of the parallel current at rational surfaces $\iota=n/m$ produce negative values of $\Omega$ because of Schwarz's inequality. In the NSTAB code the nonexistence of smooth solutions associated with these singularities is compensated for through a weak formulation of the mathematical equations.

Linear stability is hard to deal with for stellarators by numerical methods because the equilibrium problem in three dimensions does not possess smooth solutions without islands or current sheets at the resonant surfaces where the small denominators $n-\iota m$ vanish, so required differentiations are difficult to perform in a meaningful way. The spectral computations that arise involve approximating divergent Fourier series that can only be summed by filtering them in ways that are not easy to specify in advance. The problem becomes especially intractable if the nonexistence of a solution of the approximating system results in failure to drive the residuals toward zero in the numerical calculations. We circumvent this difficulty by applying the minimax theory of critical points to replace the question of stability by the question whether there are bifurcated equilibria. Thus we apply a mountain pass theorem from topology to the energy landscape which asserts that if several solutions of a conservation form of the MHD equations derived from the variational principle of magnetohydrodynamics can be found then there must be at least one that is unstable. Estimates of $\beta$ limits obtained this way tend to be significantly larger than standard predictions for ballooning modes. Similar discrepancies have been reported in other comparisons of 3D equilibrium and stability theory with ballooning calculations.

### C. Conventional tokamaks and stellarators

To study the stability of axially symmetric tokamak equilibria using the NSTAB code, it is convenient to divide the device artificially into a number $Q$ of field periods so that modes with that periodicity become isolated. We have applied this procedure to a simplified model of the International Thermonuclear Experimental Reactor (ITER) tokamak experiment proposed by the international fusion community. Let us divide the torus into identical field periods and seek in any one of them a bifurcated equilibrium different from the axially symmetric solution. At higher beta the Poincaré sections of the magnetic surfaces of such equilibria turn out to have ripple associated with known resonances at rational values $\iota=n/m$ of the rotational transform. The asymmetric ripple is evidence of nonlinear instability, and the $\beta$ limit found this way is well correlated with that predicted by linear stability theory.

Consider next the Heliac-1 (H-1) stellarator at the Australian National University, which is a heliac with three field periods that has been constructed using circular coils. The rotational transform is close to unity, the plasma has a very large helical excursion, and the separatrix is shaped in a complicated way that is typical of modern stellarators. Resolution becomes difficult for H-1 at the magnetic axis because of the large helical excursion, but we have performed convergence studies of the NSTAB code for this problem successfully. Monotonic decay over many thousands of iterations shows that long enough runs reduce the residuals to any desired degree. However, the approximation is asymptotic in the sense that truncation errors are big if too few harmonics are included in the solution, whereas too many may produce erroneous noise. After filtering of the Fourier series for the flux surface next to the magnetic axis, its shape becomes well defined. Calculations of the Mercier stability criterion predict local instability for $\beta$ near 1% in agreement with results using other codes.

We have examined runs of NSTAB for the H-1 stellarator at average $\beta$ of 2% after 35,000 iterations on a fine mesh of 44 points in the radial direction, 60 points in the poloidal direction and 48 points in the toroidal direction. Poloidal harmonics of degree 20 and toroidal harmonics of degree 16 were used in a coordinate system which rotates helically with the magnetic axis. Extensive tests performed by triggering resonant modes give no indication of nonlinear instability or of the existence of bifurcated solutions despite decidedly unstable values $\Omega<0$ for the Mercier criterion. On the other hand, cross sections of the flux surfaces display pronounced ripple consistent with resonances $\iota=1/3$ and $\iota=1/2$ just outside the range $0.35\leqslant\iota\leqslant0.37$ of the rotational transform over one field period. The observed mode is of one particular sign associated with related coefficients $B_{mn}$ in the spectrum and clashes visibly with the prescribed shape of the fixed boundary of the plasma. In the absence of free boundary calculations we interpret the incompatible ripple of the solution as evidence that an equilibrium $\beta$ limit has been exceeded so that the magnetic surfaces may break up near the separatrix. At corresponding pressures in the experiment one might look for deterioration of the confinement rather than growth in MHD activity.

We attribute the failure of the Mercier local stability criterion for H-1 at low $\beta$ to relatively large values of the terms $B_{10}$, $B_{21}$, and $B_{31}$ in the spectrum, which produce substantial parallel currents even though the resonant surfaces for these harmonics are not included within the $\iota$ range of the configuration. The parallel currents lead to perceptible shifts of the plasma, including changes in shape associated with the $m=2$, $n=1$ and $m=3$, $n=1$ modes. The apparent local instability is as much a symptom of nonexistence of the solution as it is evidence of significant global instability. This suggests that a more realistic test for $\beta$ limits in low shear stellarators similar to H-1, or in comparable advanced stellarators, is the appearance of unacceptable ripple in the magnetic surfaces calculated by the NSTAB code on fine grids. We note that the BETAS code provides a similar test, but that other methods do not.

To further validate our theory numerical studies have been made of the CHS stellarator constructed in Japan so that NSTAB calculations can be compared with the experimental measurements that are available.[11] When the vertical field is adjusted to shift the magnetic axis inward, the stability limit drops to a $\beta$ of 0.5%, which was the highest value observed in the CHS experiment for such a case. With the magnetic axis in an outward position the equilibrium and stability limit seems to be near a $\beta$ of 3%, but that conclusion is sensitive to assumptions about geometry and profiles.

We have performed NSTAB computations for a similar mathematical model of the LHD experiment (cf. Fig. 2). The code worked well despite a singularity in $p'$ at the magnetic axis for triangular pressure profiles $p=p_0(1-s^{1/2})$ shaped like observed values of the temperature. Numerical estimates of stability for poloidal mode numbers as high as $m=12$ have been made successfully, and convergence studies suggest that high modes are less dangerous than low modes. Moreover, calculations of local ballooning stability predict a $\beta$ limit decisively lower than values that have been observed in the experiment.

Recent measurements in the LHD with the radius of the magnetic axis set at $R=3.6$ m have achieved an average $\beta$ of 3.5% for stiff pressure profiles apparently associated with high shear at the edge of a relatively thick plasma.[12] In this case our calculations at first show linear instability, but nonlinear stability, of MHD modes that are too localized to destroy the equilibrium. Then good confinement in an outer rim of high shear permits access to a second stability regime when the pressure distribution is similar to triangular electron temperature profiles that have been observed.[13] For larger values of $R$ an equilibrium limit on $\beta$ is reached with magnetic surfaces deteriorating at the edge of the plasma.

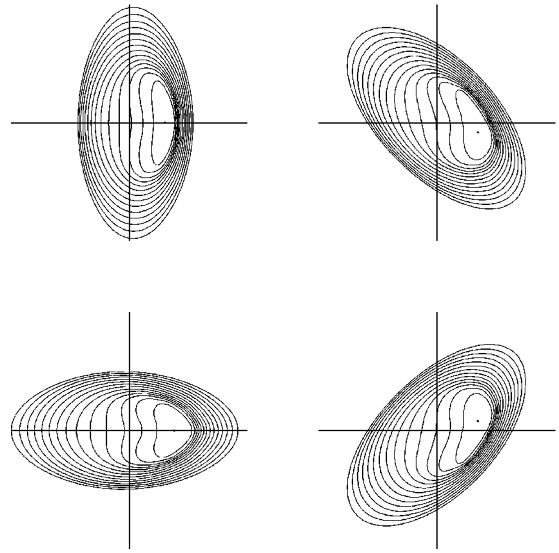

FIG. 2. Four Poincaré sections of the flux surfaces over five field periods of an LHD equilibrium with a major radius $R=3.6$ m and with a triangular pressure profile $p=p_0(1-s^{1/2})$ like temperature profiles observed in the experiment. Note the pedestal of good confinement around the edge of the plasma, which is in a second stability regime at average $\beta=0.035$, the highest value achieved thus far in the experiment.

### D. The QAS stellarator

Asymptotic expansions have been applied to exact two-dimensional solutions of the magnetostatic equations to design conventional stellarators like the LHD. They have tightly wound helical coils providing abundant shear and a good divertor, but there may be a magnetic hill in the vacuum field and the magnetic spectrum has conflicting harmonics that lead to poor transport at reactor conditions. A later generation of stellarators has been developed by running 3D computer codes like NSTAB and TRAN. They have modular coils and a larger value of the quotient $A/Q$ of the aspect ratio and the number of field periods. There is less shear, but a magnetic well provides stabilization of ideal magnetohydrodynamic modes, and the spectrum can be adjusted to give good transport at low collisionality.[14] The main feature the new designs have in common is a relatively low value of the coefficient $B_{21}$ despite a big value of the helical elongation $\Delta_{21}$ accounting for most of the rotational transform. This enables one to achieve quasihelical, quasiaxial, or quasipoloidal symmetry of the spectrum $B_{mn}$.

The Modular Helias-like Heliac (MHH) stellarator is an example of what has been achieved by numerical methods.

Line tracing based on the NESCOIL code has been applied to design its modular coils, which can be built and supported in a practical way.[15] An MHH4 configuration that was selected for an American reactor study had four field periods, a good magnetic well, low parallel current, and a rotational transform in the interval $1.0<\iota<1.25$. Optimized parameters providing simpler geometry than that of the W7-X under construction in Germany were found by imposing less stringent but perhaps more realistic stability requirements.[16]

The design of modern stellarators like the W7-X and the MHH4 is based on analysis of the Fourier expansion of $1/B^2$ in terms of renormalized flux coordinates. Its spectrum of Fourier coefficients $B_{mn}$ control most physical properties of the configuration. Each of them is directly related to the corresponding Fourier coefficient $\Delta_{mn}$ in our representation of the plasma surface. Good equilibrium, stability, and transport are achieved by reducing the size of $B_{mn}$ for selected values of the indices $m$ and $n$. That is to be accomplished so that there will be little change in the location or rotational transform of the plasma as $\beta$ increases. A test of this concept is planned in the Helically Symmetric Experiment (HSX) constructed at the University of Wisconsin, which is a quasihelically symmetric (QHS) stellarator with four field periods.[17]

We have been able to design an MHH2 stellarator with only two field periods that has the low plasma aspect ratio $A=2.5$, has good quasiaxial symmetry, and has an external magnetic field generated by just twelve modular coils. Quasiaxial symmetry signifies that the spectral coefficients $B_{mn}$ with $n\neq 0$ are relatively small, for example less than 2% of $B_{00}$. Nonlinear stability is satisfactory, and islands near a dangerous resonance at $\iota/2=1/4$ are suppressed by reducing the corresponding term $B_{41}$ in the magnetic spectrum. Good confinement at low collision frequencies is assured because two-dimensional symmetry restricts the banana orbits.

Convergence studies of bifurcated solutions of the magnetostatic equations taking into account bootstrap current show that with profiles

$$\iota=0.56-0.16s, \quad p=p_0(1-s^{1.7})^{1.3},$$

the most dangerous MHD modes of the MHH2 remain stable for $\beta$ below 5%. The shape of the magnetic surfaces indicates the place where the worst problem with stabilitry occurs. This nonlinear analysis appears to be an effective way to deal with ballooning modes in stellarators, a conclusion that is borne out by runs of the NSTAB code using spectral terms of degree as high as 36. A significant equilibrium limit on $\beta$ for the MHH2 is predicted by high-resolution NSTAB runs above 5% that display magnetic surfaces with incompatible ripple. Parameters defining the separatrix of an optimal configuration are under investigation (cf. Table I).

We are developing the MHH2 as a reactor concept with large radius 9 m, plasma radius 3 m, and a minimum gap of 1.7 m between the separatrix of the plasma and the filaments defining the modular coils, which have acceptable curvature. There is enough flexibility in the system to allow for corrections that may become necessary after more detailed studies of alpha particle confinement and the divertor are made. Monte Carlo simulations of the MHH2 reactor using the TRAN computer code show that the energy confinement time $\tau_E$ scales like $\rho_L^{-2.5}$, where $\rho_L$ is the ion gyroradius measured in units of the plasma radius. More specifically, in the case of a magnetic field of 5 T in the plasma with average temperature 12 keV and average density $1.7\times 10^{20}\,\mathrm{m}^{-3}$, we estimate that $\tau_E=3$ s. For a more conventional stellarator like the LHD the corresponding estimate of $\tau_E$ turns out to be an order of magnitude smaller so that the size of a comparable reactor becomes very large. This is because incompatible Fourier coefficients $B_{mn}$ for the conventional stellarator are so much bigger than those of a quasisymmetric helias such as the MHH2 or the HSX.

TABLE I. Coefficients $\Delta_{mn}$ defining an MHH2 stellarator for values $m=-1,0,1,2,3,4$ of the poloidal index and values $n=-1,0,1,2,3$ of the toroidal index. The spectrum of this example has a mirror term $B_{01}<0$ that raises the $\beta$ limit for nonlinear stability of ballooning modes above 5% when the configuration is run as a hybrid.

| $m\backslash n$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $-1$ | 0.190 | 0.130 | $-0.015$ | 0.000 | 0.000 |
| 0 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 |
| 1 | 0.140 | 3.000 | 0.250 | 0.050 | 0.000 |
| 2 | 0.000 | $-0.090$ | $-0.410$ | $-0.070$ | 0.000 |
| 3 | 0.000 | 0.020 | $-0.040$ | 0.080 | 0.000 |
| 4 | 0.000 | 0.015 | 0.015 | $-0.015$ | $-0.015$ |

Large values of the bootstrap current can be exploited in quasiaxially symmetric stellarators with three field periods to contribute rotational transform improving the equilibrium.[18] The QAS3 is a configuration of this kind that we have designed in collaboration with Long-Poe Ku at PPPL as a stellarator-tokamak hybrid (cf. Table II). The plasma has aspect ratio $A=4.5$ and 24 modular coils are used to produce the external field. Symmetric cross sections of the flux surfaces in runs of the NSTAB computer code establish nonlinear stability for average $\beta$ as high as 6%. A unique solution is obtained with no damaging ripple in the flux surfaces, so neither the equilibrium nor the stability $\beta$ limit is surpassed.[19] However, the issue of ballooning modes has been difficult to study because there are large gradients of the flux coordinates that make divergent Fourier series in the analysis hard to sum reliably (cf. Fig. 3). The external magnetic field generates enough rotational transform to control free boundary kink modes of the QAS3, but near the edge of the plasma there may be trouble with islands at low order resonances.[20] Performance is predicted to be optimal if net

TABLE II. Coefficients $\Delta_{mn}$ defining a QAS3 configuration for values $m=-1,0,1,2,3,4$ of the poloidal index and values $n=-1,0,1,2,3$ of the toroidal index. This is a stellarator-tokamak hybrid that depends on bootstrap current for good performance.

| $m\backslash n$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $-1$ | 0.16 | 0.09 | 0.00 | 0.00 | 0.00 |
| 0 | 0.00 | 1.00 | 0.03 | 0.00 | 0.00 |
| 1 | 0.05 | 4.50 | 0.05 | $-0.01$ | $-0.01$ |
| 2 | 0.00 | $-0.18$ | $-0.38$ | 0.02 | 0.01 |
| 3 | 0.00 | 0.09 | 0.00 | 0.06 | 0.00 |
| 4 | 0.01 | $-0.02$ | 0.02 | 0.00 | $-0.02$ |

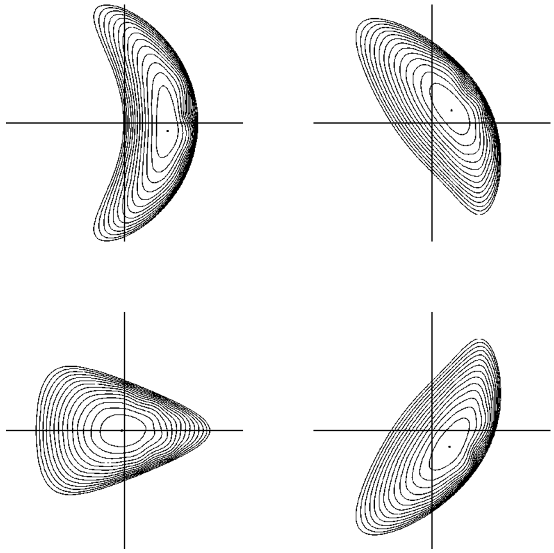

FIG. 3. Asymmetrical cross sections of a bifurcated equilibrium of the QAS3 stellarator at $\beta=0.08$ with net current putting the rotational transform in the range $0.72>\iota>0.54$. The existence of more than one equilibrium assures that there is a minimax solution at these conditions which is unstable. Notice the ballooning structure of the mode, which is localized in a region of bad curvature and decays along a magnetic line in as little as half a field period.

current is induced to keep the rotational tranform $\iota$ in a desirable range.

To make the mountain pass analysis of stability convincing, a bifurcated solution with asymmetrical flux surfaces must be exhibited that is fully converged. That has been accomplished easily at a $\beta$ of 1% for a compact configuration like one of the candidates for a stellarator experiment.[21] It has three field periods, and its aspect ratio is only $A=3.4$, but there is a substantial magnetic hill in the vacuum field when no plasma is present. More recent designs suffer from an excessive elongation of cross sections in the region of bad curvature that apparently results from pessimistic predictions of ballooning instability. Another interesting proposal has been made for a concept exploration experiment based on a quasiomnigenous stellarator (QOS) not unlike the MHH2. To obtain this one adjusts the mirror term $\Delta_{01}$ to produce large negative values of $B_{01}$ so the coefficient $B_{10}$ decreases, the bootstrap current falls off, and ballooning stability improves.

The MHH2 and QAS3 stellarators have good equilibrium, stability, and transport, and attractive sets of modular coils have been designed for them. Substantial progress has been made on some of the mathematical problems in stellarator theory, and we have been able to optimize these configurations for a proof of principle experiment to demonstrate that the concept is competitive with tokamaks as a toroidal confinement system for magnetic fusion. However, important technological issues must be addressed to construct a successful compact stellarator with modular coils delivering an accurate external magnetic field. It is important to find out whether the quasiaxially symmetric configurations perform well as stellarator-tokamak hybrids. In that connection, as a hybrid the version of the MHH2 defined in Table I is predicted to operate well when the rotational transform is adjusted to come out in the range $0.6>\iota>0.4$.

### E. Coils and islands

After fixed boundary calculations of equilibrium and stability have been used to design an optimal stellarator configuration, it becomes necessary to find coils that will produce a compatible external magnetic field. In practice this is accomplished by looking for a vacuum field given by the Biot–Savart law

$$\mathbf{B}=\nabla\times\int\int\frac{\nabla\varphi\times\mathbf{N}}{r}d\sigma,$$

that has a flux surface identical with the boundary of the optimal plasma. The integration is to be performed over a suitable control surface specified by a formula like the one that appears in the equilibrium code, and

$$\varphi=v+\sum\varphi_{mn}\sin(mu-nv),$$

is a distribution of surface current whose level curves provide filaments defining a set of modular coils for the stellarator. The mathematical problem of determining the coils is not well posed because it involves an analytic continuation and approximation of **B** related to Runge's theorem in complex analysis, which concerns the representation of an analytic function by infinite series of polynomials in simply connected regions. What this means is that the coils turn out to be too complicated to construct in an effective fashion unless care is exercised in selecting the trigonometric polynomials that are employed.

We have written a line tracing code called NWIND that can be run to calculate the Fourier coefficients $\Delta_{mn}$ of the last closed flux surface defined by the Biot–Savart law and see whether it furnishes an adequate approximation to the original equilibrium at zero $\beta$. This tool enables us to readjust the definition of the coils systematically and arrive at a better fit. Thus there is also a well posed, but more tedious, mathematical approach to the problem of designing coils. Good results are obtained when one uses just coefficients $\varphi_{mn}$ with the indices $m$ and $n$ of the shape factors $\Delta_{mn}$ that occur. Tables I and II show how the indices are chosen in practical cases.

In a compact stellarator the rotational transform over each field period is relatively large, so low order resonances can produce undesirable magnetic islands even if there are no field errors. Such an island can be suppressed by requiring at any dangerous rational surface where $\iota=n/m$ that the corresponding term $B_{mn}$ in the magnetic spectrum vanishes so the parallel current does not become singular. It is not easy to accomplish that, so a certain amount of trial and error is called for. However, the coefficient $\varphi_{mn}$ largely controls the corresponding factor $\Delta_{mn}$ defining the shape of the plasma, and that in turn controls the spectral term $B_{mn}$ in less resonant cases, which makes the design problem manageable. We succeed by filtering the Fourier series in the solution to eliminate irrelevant harmonics and by avoiding irregularities in the geometry of the coils and the plasma.

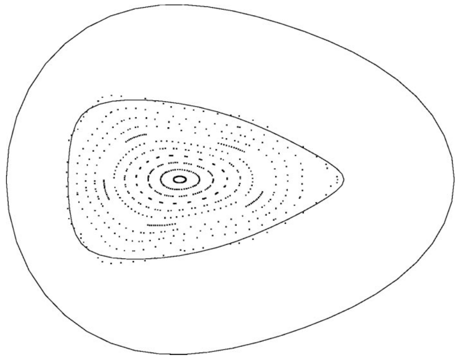

FIG. 4. Poincaré section of the flux surfaces of an MHH2 stellarator in a vacuum magnetic field given by the Biot–Savart law. The outermost curve shows where the control surface is on which coil filaments are located, and the inner curve is the free boundary of an equilibrium calculated by the NSTAB code. There is an $m=5,n=1$ island inside the plasma that has been largely suppressed through choice of the coefficient $\varphi_{51}$ in the Fourier series for the surface current whose level curves define the modular coils.

For an MHH2 stellarator with two field periods and aspect ratio $A=2.5$ we have applied the method just described to find a set of 12 only moderately twisted modular coils (cf. Fig. 1). There is adequate space between them, and the minimum distance between the plasma and the control surface is half the plasma radius. At zero $\beta$ the rotational transform over one field period lies in the interval $1/4>\iota>1/5$. This means that the most dangerous resonance is $m=4, n=1$, and fortunately the corresponding coefficient $B_{41}$ is relatively small. That term in the magnetic spectrum remains less than 0.3% of $B_{00}$ for $\beta>0$ and for profiles of $\iota$ that take into account the effect of bootstrap current.[22] Both equilibrium and line tracing calculations suggest that we should be able to control the corresponding island through small adjustments of the coefficient $\varphi_{41}$ in the Biot–Savart law. In an experiment space should be provided to allow for auxiliary coils generating this harmonic.

The term $B_{51}$ is larger and more troublesome, but fortunately $\iota$ is expected to rise above 1/5 in the plasma as the bootstrap current increases with $\beta$. Figure 4 shows how an $m=5, n=1$ island has been largely suppressed inside the plasma by our choice of $\varphi_{51}$ for another MHH2 stellarator in a vacuum field defined by the Biot–Savart law with $\iota$ crossing the resonant value 1/5. After that has been accomplished any difficulties with the corresponding island in an actual experiment should be no worse than those in a more conventional stellarator that are encountered when there are field errors.

To avoid undesirable ripple in the magnetic field of the QAS3 stellarator with three field periods defined in Table II, we selected 24 modular coils for it that are adequately spaced and not excessively twisted. This case has reversed shear, and the rotational transform over one field period in a vacuum rises to between 1/7 and 1/6 at the separatrix. Corresponding islands can be reduced by adjusting of the coefficients $\varphi_{71}$ and $\varphi_{61}$, but when the bootstrap current raises the rotational transform the problem with magnetic surfaces near the edge of the plasma might become more difficult. In practice one should allow for small auxiliary coils that could be added in an experiment to cancel bad effects from bootstrap current or field errors. Vertical field coils can be included to compensate for increasing $\beta$, just as in a torsatron. Fortunately the modular coils for the QAS3 stellarator are less twisted than those required in more conventional configurations. Finally, induced current can also be used to avoid dangerous resonances.

Quasisymmetric stellarators should be considered as an alternate concept for the large ITER project that is being planned. The topology of the coils is like that of a tokamak, but the rotational transform does not vanish at the separatrix to complicate problems with edge physics. In this context it is natural to view the MHH2 and the QAS3 as advanced tokamaks. To make more progress it will be necessary to construct an experiment adequate to validate the theory.

## III. MONTE CARLO SIMULATION OF TRANSPORT

### A. Introduction

An effective method for the analysis of transport in stellarators and tokamaks is to track particle orbits subject to a random walk in velocity space. The orbits are found by solving guiding center differential equations, and we employ a test particle model from kinetic theory to evaluate the confinement time. This involves putting a sample particle in the plasma and calculating how long it takes to leave. Because the effect of a single particle on macroscopic quantities is negligible, the electromagnetic field and the flow of the background are taken to be fixed. To estimate the influence of random collisions we study the motion of the test particles statistically using a distribution function that satisfies a simplified drift kinetic equation. The collisions are represented by a second-order elliptic partial differential operator acting with respect to velocity coordinates, as in a Fokker–Planck equation. Solutions should decay to the normal distribution, or Maxwellian, according to the central limit theorem. Details about the mathematics of this approach differ from statistical mechanics as it is usually formulated in plasma physics.[23]

The test particle model can be implemented numerically by the Monte Carlo method. The test particles move on drift surfaces, but march back and forth among them following a random walk that simulates collisions. Any particle that reaches the boundary of the plasma escapes, so the distribution function vanishes there. This process defines a solution of the drift kinetic equation that decays exponentially at a rate specifying the particle confinement time.

Because of a singular perturbation of Poisson's equation that is associated with small Debye length and neglect of the displacement current in our transport theory, the electric potential should be determined by the requirement of charge neutrality rather than from equations of motion represented by Ohm's law. Resulting oscillations of the electric field along lines of magnetic force serve in this model to explain anomalous transport of electrons. Numerical simulations that assume a simpler radial electric field give a confinement time for the electrons significantly higher than that for the ions. That leads to different rates of decay and hence to different sources for the ions and electrons to maintain a steady state.

The impossibility of producing these different sources by injecting neutral particles may explain the origin of turbulence, which is presumably triggered by the displacement current as it enforces quasineutrality.

We perform transport calculations using the TRAN Monte Carlo code. Exponential rates of decay of expected values of carefully selected integrals of the solution of the drift kinetic equation provide estimates of the particle confinement times $\tau_i$ and $\tau_e$ for the ions and electrons separately. Thus the code enables us to study electron transport efficiently. It is important to exploit this capability, which enables us to achieve good agreement with experimental data by combining the ion and the electron calculations with a mechanism to enforce quasineutrality and evaluate the electric field.

We use the MHD equations to determine the structure of the magnetic field in the plasma, which allows us to express the guiding center differential equations in terms of the invariant flux coordinates $s$, $\theta$, and $\phi$. If we renormalize $\theta$ and $\phi$ so that they become poloidal and toroidal angles on each flux surface, then all we need to integrate the equations is a knowledge of the Fourier representation of the magnetic-field strength, which characterizes the spectrum.[24]

### B. Quasineutrality

Including the displacement current in Maxwell's equations makes them consistent with conservation of charge. In the relativistic formulation we have

$$\lambda_D^2 \mathbf{E}_t = \nabla \times \mathbf{B} - \mathbf{J},$$

where $\lambda_D$ is the Debye length and $\mathbf{E}$ is the electric field. The equation for the displacement current is a relation between the electromagnetic fields and the current density

$$\mathbf{J} = q n_i \mathbf{u}_i - q n_e \mathbf{u}_e,$$

expressed in terms of the number densities $n_i$ and $n_e$ and fluid velocities $\mathbf{u}_i$ and $\mathbf{u}_e$ of the ions and the electrons. Since the Debye length is small, it is customary in plasma physics to neglect the displacement current. Then the three components of the magnetic field must solve the four simultaneous equations

$$\nabla \times \mathbf{B} = \mathbf{J}, \quad \nabla \cdot \mathbf{B} = 0.$$

A solution cannot be found unless the compatibility requirement of quasineutrality $\nabla \cdot \mathbf{J} = 0$ is imposed. Hence in any analysis involving $\mathbf{B}$ the plasma is presumed to satisfy that condition.

The displacement current is neglected because the factor $\lambda_D^2$ is small, not because the time derivative $\mathbf{E}_t$ is zero. If it were retained in Maxwell's equations, it would generate fluctuations of the electric potential $\Phi$ along the magnetic lines in three dimensions. These fluctuations are observed as plasma oscillations and turbulence that produce significant transport. Time-dependent behavior of that kind ought to be simulated in our static Monte Carlo model. Also, it becomes natural to consider quasineutrality as the right equation for the electric potential because of the role played by leaving out the Laplacian in Poisson's equation

$$\lambda_D^2 \Delta \Phi = n_e - n_i,$$

where $\Phi$ is measured in units of the temperature. The theory of singular perturbations implies that when the term on the left is neglected, the resulting requirement of quasineutrality

$$n_e(\Phi) - n_i(\Phi) = 0,$$

is still what determines the electric field, even though $\Phi$ no longer appears explicitly. This is the mechanism that will provide the desired oscillations of $\Phi$ along the magnetic lines in our model.

Often in plasma physics one measures the flow $\mathbf{u}$ and finds the electric field from Ohm's law

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{J},$$

which is applied to establish its radial character. If the resistivity $\eta$ is neglected, integration of the magnetic differential equation $\mathbf{B} \cdot \nabla \Phi = 0$ suggests that $\Phi$ is a function of the toroidal flux $s$ alone. This is a dubious approximation based on an ill posed problem in which the right-hand side does not even vanish, and it is probably responsible for anomalies in the conventional theory of transport. The electric field is dominated by a radial term, but small oscillations of the electrostatic potential within the magnetic flux surfaces may have the same effect as the observed fluctuations to which anomalous transport is usually attributed. A plausible mechanism to enforce the quasineutrality requirement $n_e = n_i$ is called for to model such oscillations and provide a time-averaged simulation of more complicated physical phenomena.

Based on the considerations just described, we replace Ohm's law by quasineutrality as the rule to determine the electric potential. Experimentalists are accustomed to observing impurity flow and then estimating the radial electric field from Ohm's law, which is a reasonable thing to do, but it is wrong to calculate the much smaller parallel component of $\mathbf{E}$ the same way in a detailed study of transport.

To calculate the oscillations of $\Phi$ we solve the quasineutrality and ambipolarity equations $n_e = n_i$, $\tau_e = \tau_i$ iteratively. The mechanism implemented in the TRAN code to achieve this requires that Fourier coefficients of the densities $n_i$ and $n_e$ of the ions and electrons coincide. The electric potential $\Phi$ is expanded as a Fourier series

$$\Phi = \sum P_{mn} \cos(m\theta - n\phi),$$

in the invariant poloidal and toroidal angles $\theta$ and $\phi$ with coefficients $P_{mn}$ that depend on the flux $s$. The charge separation is expressed as a similar Fourier series

$$n_e - n_i = \sum C_{mn} \cos(m\theta - n\phi),$$

after dividing out the total numbers of electrons and ions. Calculation of low order Fourier coefficients $C_{mn}$ is accomplished by using the Monte Carlo method to estimate corresponding expected values of trigonometric functions that are statistically significant.

We propose to determine the coefficients $P_{mn}$ so that $C_{mn} = 0$. Because of the way charge separation gives rise to

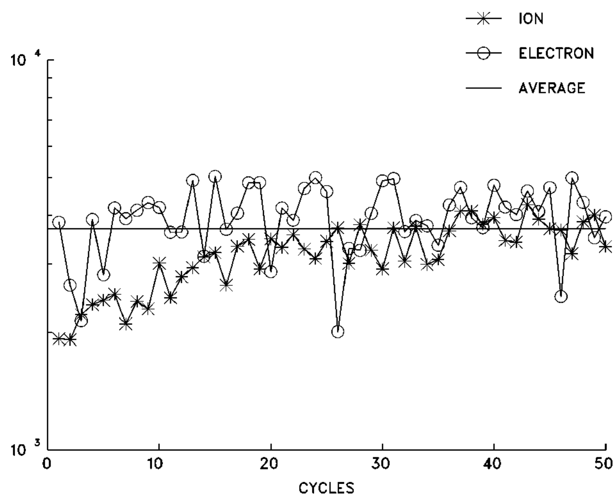


FIG. 5. Iterations converging to quasineutrality in a Monte Carlo computation of the energy confinement time $\tau_E$, measured in milliseconds, for an MHH2 stellarator of aspect ratio $A=3$ at reactor conditions with average $T=12$ keV, $n=1.7\times10^{20}\,\mathrm{m}^{-3}$, and $B=5$ T. The magnetic spectrum has excellent axial symmetry, but we have superimposed a mirror term that improves transport by driving the radial electric field to a potential level four times the temperature.

electrostatic restoring forces, it is reasonable to expect wells and hills in the electric potential to compensate for similar oscillations in $n_e-n_i$. This suggests that $P_{mn}$ has a dominant effect on the corresponding coefficent $C_{mn}$, so in practice we find a numerical solution by truncating the two Fourier series for $\Phi$ and $n_e-n_i$ in the same way. For a suitable relaxation factor $\epsilon$, a convergent iteration scheme driving $C_{mn}$ towards zero takes the form

$$P_{mn}^{l+1}=P_{mn}^{l}+\epsilon C_{mn}^{l}.$$

Numerical calculations using this algorithm show that relatively small values of $P_{mn}$ with magnitude only a few percent of the temperature may account for the anomalous transport of electrons, for they reduce $\tau_e$ much more than $\tau_i$.

The method described thus far is insufficient because $P_{00}$ is not yet determined, for the average value of $C_{00}$ has been normalized to vanish. One of the advantages of the TRAN code is its effectiveness in computing both the ion and the electron confinement times, so we solve the ambipolarity equation $\tau_e=\tau_i$ to find a physically relevant value of the radial potential $P_{00}$. Stellarator calculations have shown that an increase in the size of $P_{00}$ increases the ion confinement time while leaving the electron confinement time more or less unchanged. Hence the iteration

$$P_{00}^{l+1}=\left(1-\epsilon\log\frac{\tau_i}{\tau_e}\right)P_{00}^{l},$$

completes the algorithm to solve for quasineutrality and ambipolarity in stellarators. It yields both the expected ion root and the expected electron root. Figure 5 displays a calculation illustrating this procedure in which good resolution was achieved by tracking collections of 1024 test particles. The result gives an indication of how turbulence and anomalous transport may be controlled in quasisymmetric stellarators.

For tokamaks the dependence of confinement times on $P_{00}$ is less pronounced because of axial symmetry, so we simply fix the radial term at some plausible value. On the other hand, computations of 3D bifurcated tokamak equilibria show that an increase in the helical excursion $\Delta_{11}$ of the magnetic axis can reduce the electron confinement time without affecting the ion confinement time very much. Similar results can be achieved by increasing the size of other resonant MHD modes. Therefore, to study tokamak transport, the axially symmetric coefficients $B_{m0}$ in the spectrum are obtained from an equilibrium calculation, but small 3D terms $B_{mn}$ with $n\neq0$ are allowed to appear, too. We introduce a family of solutions parametrized by the size of the 3D perturbation and iterate on that amplitude to solve the ambipolarity equation $\tau_e=\tau_i$. Increasing the magnitude of the perturbation reduces $\tau_e$ without changing $\tau_i$ so much, but decreasing it has the opposite effect. The method converges well in practice and gives results in agreement with experiment. However, this 3D simulation of transport in tokamaks is more speculative than the algorithm we have described for stellarators because of the way it makes use of bifurcated equilibria.

The asymmetrical perturbations of **E** and **B** that we have described provide a 3D simulation of turbulence associated with microinstabilities and drift waves. The quasineutrality algorithm gives results for the confinement time that are relatively insensitive to the choice of a specific MHD mode altering the form of $B$ in three dimensions. We have found that the method converges adequately, and calculations have been run for tokamaks to validate the model through comparison with experimental measurements.

### C. Test particle equations

In tokamaks and stellarators the mean free paths of the ions and electrons are hundreds of times longer than the radius of the device. The gyroradius $mv_\perp/qB$ is small compared with the plasma radius, so orbits can be tracked by solving guiding center equations of the form

$$\dot{\mathbf{x}}=\rho_\parallel[\mathbf{B}+\nabla\times(\rho_\parallel\mathbf{B})].$$

Here the parallel gyroradius $\rho_\parallel=mv_\parallel/qB$ is of the same order of magnitude as the gyroradius. It is determined from the conservation of energy

$$W=\frac{q^2B^2\rho_\parallel^2}{2m}+\mu B+q\Phi,$$

where the magnetic moment $\mu=mv_\perp^2/2B$ is an adiabatic invariant that is constant along trajectories between collisions. The guiding center system can be transformed into simpler ordinary differential equations in terms of the flux coordinates $s$, $\theta$, and $\phi$ that just involve $1/B^2$, which explains why two-dimensional symmetry of the magnetic spectrum $B_{mn}$ is important for transport.

The Monte Carlo method of calculating transport is most easily formulated in terms of a drift kinetic equation

$$\frac{\partial f}{\partial t}+\rho_{\parallel}[\mathbf{B}+\nabla\times(\rho_{\parallel}\mathbf{B})]\cdot\nabla f$$
$$=\nabla\cdot[\nu\, e^{-m(\mathbf{v}-\mathbf{u})^2/2T}\nabla e^{m(\mathbf{v}-\mathbf{u})^2/2T} f],$$

for the distribution function $f$ of each species of test particle. The operator $\nabla$ acts on space coordinates $\mathbf{x}$ on the left, but on velocity coordinates $\mathbf{v}$ on the right. The collision frequency $\nu$ and the temperature $T$ are given functions of $|\mathbf{v}-\mathbf{u}|$ and the flux $s$, respectively. The term in square brackets on the left is the guiding center velocity, and the term on the right is the collision operator. There are actually two different equations to solve, one for the ions and one for the electrons, distinguished by their respective gyroradii, collision frequencies, and temperatures, so it is possible to compute independently both the ion confinement time and the electron confinement time. The collison operator is designed to drive $f$ toward a normal distribution shifted by the fixed fluid velocity $\mathbf{u}$ of the background. It conserves mass, but not necessarily momentum, which is a property that depends on the choice of $\mathbf{u}$.

Exponential decay of $f$ can be established by the method of separation of variables because $t$ does not appear explicitly in the drift kinetic equation. Let us look for solutions of the form $f=e^{-\lambda t}F$ in which $F$ does not depend on time. Substitution of this relation into the drift kinetic equation and differentiation of products on the right leads to the eigenvalue problem

$$\rho_{\parallel}[\mathbf{B}+\nabla\times(\rho_{\parallel}\mathbf{B})]\cdot\nabla F=\nabla\cdot\left[\nu\left(\nabla F+\frac{m(\mathbf{v}-\mathbf{u})}{T}F\right)\right]+\lambda F.$$

If the smallest eigenvalue $\lambda$ is positive, then solutions of the drift kinetic equation decay exponentially, and the rate of decay is dominated by that eigenvalue. The particle confinement time $\tau=1/\lambda$ defined to be the inverse of the smallest eigenvalue is a measure of how long it takes for test particles to leave the plasma.

If we identify the term $\lambda F$ on the right-hand side of the eigenvalue equation as a source $S$, then for the lowest eigenvalue we have $S=F/\tau$. This source specifies one way particles can be injected into the plasma to maintain a steady state. For a reasonable choice of norm, the Rayleigh quotient

$$\tau=\|F\|/\|S\|,$$

gives a representation of the confinement time analogous to the ratio of energy divided by power that is introduced as the measured value of energy confinement time in experiments. Note that there are two distinct sources, one for the ions and another for the electrons, so when the confinement times are different, the sources are, too. Because only neutral particles can be injected into the plasma we see that steady solutions with unequal sources are not realistic in an analysis of transport, which means they become irrelevant when the electron confinement time is an order of magnitude larger than the ion confinement time.

The TRAN code determines the particle confinement time $\tau$ rather than the energy confinement time $\tau_E$. For comparison with experiments we need a relationship between these two quantities, and the semiempirical formula $\tau_E=\tau/3B$ seems to work well in practice, where the division by $B$ arises from a normalization of time by the ion gyrofrequency. In the TRAN code particle confinement time is evaluated by estimating the exponential rate of decay of functionals calculated by the Monte Carlo method. Numerical experiments have shown that good estimates for both the ion and the electron confinement times are obtained using the functional

$$\sum\sum\cos\left(\frac{s_{jk}-\sigma_k}{1-\sigma_k s_{jk}}\frac{\pi}{2}\right)=Ae^{-t/\tau}+\cdots,$$

where $s_{jk}$ stands for the coordinate $s$ of the $j$th particle launched from the flux surface $s=\sigma_k$. The key step in our approach is to actually perform the electron calculation, and this formula makes that possible.

Computation of confinement times and of the Fourier coefficients of the charge separation involves multiple integrals in a five-dimensional space described by the variables $s$, $\theta$, $\phi$, $W$, and $\mu$. To do the integrations we solve the drift kinetic equation for the particle distribution function $f$, which depends on these five variables plus the time $t$ and the sign of the parallel velocity. The Monte Carlo method is an effective tool for computations in space of such high dimension.

The algorithm to solve the drift kinetic equation that has been implemented in the TRAN code is a split time method which alternates between numerical calculations of guiding center orbits in flux coordinates and applications of a random walk in velocity space modeling the collision operator. This operator involves a Maxwellian shifted by the flow velocity $\mathbf{u}$, and the collision frequency is a function of $|\mathbf{v}-\mathbf{u}|$, so it is convenient to translate the coordinate system by $\mathbf{u}$ before applying the random walk. The formulas for the transformation of coordinates show that when $\mathbf{u}$ is small compared with the thermal speed the changes it makes in the energy $W$ and in the magnetic moment $\mu$ during the random walk are small. Therefore, $\mathbf{u}$ has little effect on the collision operator, in which conservation of momentum should play a secondary role, since it could be enforced by a self-consistent choice of $\mathbf{u}$.

The principal parameters in the collision operator are the temperature and the collision frequency. Runs of the TRAN code provide numerical evidence that other features are less important. In order to study this matter systematically we have modified the collision operator by altering the Maxwellian and we have observed how that influences transport. Numerical calculations establish that a variety of changes that were made have little effect.[25] There is only secondary variation of the confinement times of the ions and electrons with perturbations of the flow velocity $\mathbf{u}$ in the collision operator, so that could not compensate for failure of quasineutrality. In the test particle model the electromagnetic field and the flow velocity are assigned, so test particles scatter against a fixed background and conservation of momentum is dependent on the flow, which can safely be neglected in the computation.

The physics of our model would be more consistent if we chose the flow velocity $\mathbf{u}$ to ensure conservation of momentum in the collision operator, just as we find the electric potential $\Phi$ by imposing quasineutrality. However, that would lead to a much more elaborate computation. Our numerical studies show that it would not affect the results significantly, so we are justified in simply putting $\mathbf{u}=0$. On the other hand, the quasineutrality algorithm for $\Phi$ converges well and seems to account for anomalous transport successfully. Thus the final formulation of the method is based on both theoretical and practical considerations. Perhaps the most important conclusion is that it is wrong to think that requiring conservation of momentum could by itself lead to quasineutrality. After the radial electric field has been determined by quasineutrality, we could use conservation of momentum in the form of Ohm's law to find the flow, but that would still have a negligible effect on the confinement time through its appearance in the collision operator.

### D. Comparison with experiment

In the TRAN code the guiding center ordinary differential equations are solved by a fourth order Runge–Kutta method in which the step size is chosen so that well defined drift surfaces can be observed in Poincaré sections. Extensive runs have been made to verify that the confinement times $\tau_i$ and $\tau_e$ and the relevant Fourier coefficients $C_{mn}$ of the charge separation are statistically significant. We have analyzed the results to show that the method does not have unacceptable numerical errors and that the quasineutrality algorithm works for low collision frequencies like those in a reactor. Confinement times computed by this algorithm compare favorably with experimental data obtained from Tokamak Fusion Test Reactor (TFTR) supershots at PPPL (cf. Ref. 3).

We have compared numerical results from the TRAN code with measurements from the Joint European Tokamak (JET) experiment located in Great Britain. Experimental parameters modeling JET were plasma major radius $R=3$ m, horizontal plasma minor radius $a=1$ m, and vertical plasma minor radius $b=1.5$ m. Average values of the particle density and the temperature were $n=4\times10^{19}\,\mathrm{m}^{-3}$ and $T=4$ keV. The average magnetic-field strength was $B=3$ T, the effective charge number was $Z=2$, and the energy confinement time was $\tau_E=260$ ms. We included both axially symmetric Fourier modes and 3D modes with $m\leqslant4$ and $n\leqslant4$ that are resonant inside the plasma for our calculations using the TRAN code, and the radial electric potential was set to $-1.5$ times the temperature. In an initial run the 3D modes of the electromagnetic field were put equal to zero, so the electron confinement time became much larger than the ion confinement time. A second run showed how 3D perturbations of the fields affect transport by reducing the electron confinement time more than the ion confinement time, bringing both down to a realistic level. Iterations were performed to achieve quasineutrality, and then the confinement times of the ions and electrons reached good agreement with the experimental data.

These computations may explain why two-dimensional models of transport that do not include a mechanism enforcing quasineutrality fail to simulate experimental data. The Monte Carlo method takes into account the complex geometry of the drift surfaces, so it can describe transport that is dominated by turbulence and banana orbits. If in the calculations one can find realistic 3D perturbations of $\Phi$ and $1/B^2$ that suffice to enforce quasineutrality, then the numerical results agree better with observations.

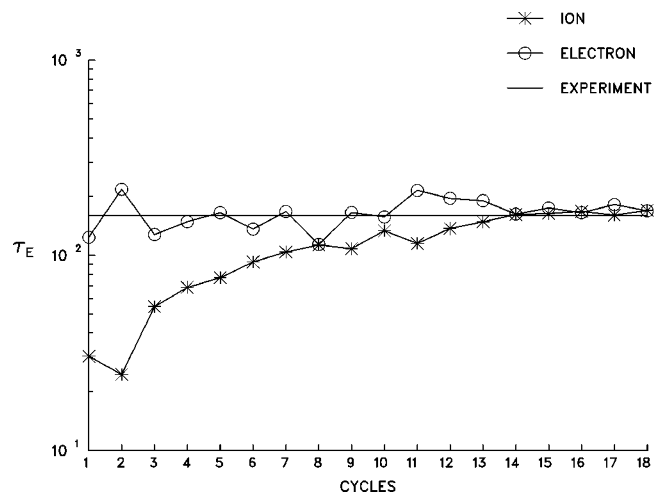


FIG. 6. Calculation of the energy confinement time $\tau_E$ in milliseconds for an NBI shot of the LHD experiment using 18 iterations on the electric potential $\Phi$ to achieve quasineutrality. Oscillations of $\Phi$ along the magnetic lines model turbulence and anomalous transport well enough so that there is satisfactory agreement with the measured value, discrediting the faulty assumption $\Phi=\Phi(s)$ that the potential is constant on flux surfaces.

Our implementation of the quasineutrality algorithm is only effective at low-collision frequencies such that a few early harmonics $P_{mn}$ suffice to approximate the electric potential $\Phi$. Therefore, we could only use tokamak data in our first comparisons of the theory with experiment, and we had to model turbulence by introducing bifurcated equilibria whose physical relevance is not universally recognized. Recently measurements of confinement time for the LHD stellarator in Japan have been performed that are at the required level of collisionality, so a more satisfactory comparison becomes feasible.[26] More specifically, we have run the TRAN code to model an observation with peak temperature 3 keV and density $1.5\times10^{19}\,\mathrm{m}^{-3}$. The central magnetic field was 2.8 T, the radius of the magnetic axis was 3.6 m, and the small radius of the plasma was 0.65 m. Both the experimental measurements and the mathematical computations gave the value $\tau_E=160$ ms for the energy confinement time (cf. Fig. 6). The margin of error is satisfactory, but depends on an estimate near 2 of the effective charge number $Z$, which occurs in formulas for the collision frequency. This is a preliminary result calling for more careful theoretical analysis of new LHD data that are becoming available.

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